Stochastic Processes in Natural Sciences – 6 November 2026. Markov processes

Problem 1: The telegraph process A process takes the values $X = \pm 1$. It models burst noise (popcorn noise, random telegraph signal) in electronic devices. The sign flips at random times: the number N of flips in any time interval of length t is Poisson distributed,

$$P(N = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!},$$

and the numbers of flips in non-overlapping intervals are independent. First take $X(0) = 1$.

(a) Find $\langle X(t) \rangle$. How quickly does the system lose memory of its initial state? *Hint:* Find the probability that Poisson process produces an even or odd number of flips in time τ.

(b) Compute the autocorrelation function $\kappa(t_1, t_2)$. *Hint:* for $t_2 \geq t_1$, write $X(t_1)X(t_2)$ in terms of the number of flips between t_1 and t_2.

(c) Is the process stationary? What happens at long times? Now let $X(0) = \pm 1$ with probability $\frac{1}{2}$ each, independently of the flips. Repeat (a) and (b).

(d) Find the correlation time and the spectral density $S(\omega)$ of the stationary telegraph process.

Problem 2: The telegraph process as a Markov process The telegraph process is a Markov process, fully characterised by writing, for $t \geq t'$,

$$P_1(x,t) = \tfrac{1}{2}\left(\delta_{x,1} + \delta_{x,-1}\right),$$
$$P_{1|1}(x,t\,|\,x',t') = \tfrac{1}{2}\left[1 + e^{-2\gamma(t-t')}\right]\delta_{x,x'} + \tfrac{1}{2}\left[1 - e^{-2\gamma(t-t')}\right]\delta_{x,-x'}.$$

(a) Check that $P_1 \geq 0$, $P_{1|1} \geq 0$, $\sum_x P_1(x,t) = 1$ and $\sum_x P_{1|1}(x,t\,|\,x',t') = 1$. What is $P_{1|1}$ at $t = t'$?

(b) Show that P_1 is stationary: $\sum_{x'} P_{1|1}(x,t\,|\,x',t')\,P_1(x',t') = P_1(x,t)$.

(c) Show that for $x, x' \in \{-1, 1\}$ the transition probability can be written compactly as

$$P_{1|1}(x,t\,|\,x',t') = \tfrac{1}{2}\left[1 + xx'\,e^{-2\gamma(t-t')}\right],$$

and use this form to verify the CKS equation.

(d) Compute $\kappa(\tau)$ from P_1 and $P_{1|1}$. Compare with Problem 1: how is γ related to λ?

Problem 3: Markovian or not? A particle moves freely, $Y(t) = X_0 + Vt$, where X_0 and V are independent Gaussian variables with zero mean and variances $\sigma_0^2, \sigma_v^2 > 0$. Let $t_1 < t_2 < t_3$.

(a) Without calculation, find $P_{1|2}(y_3, t_3\,|\,y_1, t_1; y_2, t_2)$.

(b) Argue that $P_{1|1}(y_3, t_3\,|\,y_2, t_2)$ is not a delta function. Is $Y(t)$ a Markov process?

(c) Show that the two-component process $\big(Y(t), V\big)$ is Markovian, and write down its transition probability.

Problem 4: The Wiener process Consider the homogeneous process with transition probability

$$T_\tau(y_2|y_1) = \frac{1}{\sqrt{4\pi D\tau}} \exp\!\left(-\frac{(y_2 - y_1)^2}{4D\tau}\right), \qquad \tau > 0.$$

(a) Check that $\int T_\tau(y_2|y_1)\,dy_2 = 1$ and that $T_\tau(y_2|y_1) \to \delta(y_2 - y_1)$ as $\tau \to 0$.

(b) Verify the CKS equation. *Hint:* for fixed y_1, the right-hand side of CKS is the density of y_1 plus two independent Gaussian increments. Use characteristic functions.

(c) The 1D random walk we discussed before is a discrete analogue, with $T_N(M|M') = p_{M-M'}(N)$. Use the generating function $f(N, z)$ to show that it satisfies the discrete CKS equation

$$p_M(N_1 + N_2) = \sum_K p_{M-K}(N_2)\, p_K(N_1).$$

(d) Does the Wiener process have a stationary distribution P_1, i.e. one satisfying

$$\int T_t(y|y')\, P_1(y')\, dy' = P_1(y) \quad \text{for all } t?$$

Hint: how does the variance of Y change in time?