## Sketch of the solution: (2021/22)

- write down the master equation (3 pts)

$$\dot{p}_n(t) = \gamma(N-n+1)\,p_{n-1}(t) - \gamma(N-n)\,p_n(t) - \alpha n\,p_n(t) + \alpha(n+1)\,p_{n+1}(t) \qquad n=1,\dots,N-1$$

special cases

a) $\dot{p}_0(t) = -\gamma p_0(t) + \alpha p_1(t)$

consistent with the general case if we set $p_{-1}(t)=0$

also no probability current from $p_0(t)$ to $p_{-1}(t)$

b) $\dot{p}_N(t) = \gamma p_{N-1}(t) - \alpha N p_N(t)$

consistent with the general case if we set $p_{N+1}(t)=0$

again no current from $p_N(t)$ to $p_{N+1}(t)$.

Full equation

$$\dot{p}_n(t) = \gamma(N-n+1)\,p_{n-1}(t) - \gamma(N-n)\,p_n(t) - \alpha n\,p_n(t) + \alpha(n+1)\,p_{n+1}(t)$$

with $p_{-1}(t) = p_{N+1}(t) = 0$.

- stationary solution (4 pts)

with $\mathbb{E}$ operator:

$$\dot{p}_n(t) = \gamma\left(\mathbb{E}^{-1}-1\right)(N-n)\,p_n(t) + \alpha\left(\mathbb{E}-1\right) n\, p_n(t)$$

stationary solution: $\dot{p}_n^s(t) = 0$. This gives

$$\alpha\left(\mathbb{E}-1\right)\left(n p_n^s - \frac{\gamma}{\alpha}\mathbb{E}^{-1}(N-n)\,p_n^s\right) = 0$$

stationary solution depends only on the ratio $\gamma/\alpha$

and

$$n p_n^s = \frac{\gamma}{\alpha}(N-n+1)\,p_{n-1}^s$$

or

$$p_n^s = \left(\frac{\gamma}{\alpha}\right)\frac{N-n+1}{n}\,p_{n-1} = \left(\frac{\gamma}{\alpha}\right)^2\frac{(N-n+1)(N-n+2)}{n(n-1)}\,p_{n-2}^s$$

etc.

$$p_n^s = \left(\frac{\gamma}{\alpha}\right)^n\frac{N!}{n!\,(N-n)!}\,p_0^s = \left(\frac{\gamma}{\alpha}\right)^n\binom{N}{n}p_0^s$$

$$1 = \sum_{n=0}^{N} p_n^s = p_0^s\sum_{n=0}^{N}\left(\frac{\gamma}{\alpha}\right)^n\binom{N}{n} = p_0^s\left(1+\frac{\gamma}{\alpha}\right)^N$$

Hence $$p_n^s = (\alpha+\gamma)^{-N}\cdot\gamma^n\,\alpha^{N-n}\binom{N}{n}.$$

- at thermal equilibrium (3 pts)

$$\frac{N_1}{V_1} = \frac{N_2}{V_2} \quad \text{or} \quad \frac{N_1}{N_2} = \frac{V_1}{V_2}$$

on the other hand

$$N_1 = \langle n \rangle, \quad N_2 = N - \langle n \rangle$$

where

$$\langle n \rangle = \sum_{n=0}^{N} n p_n^s = \sum_{n=0}^{\infty} n \cdot p_0^s \cdot \left(\frac{\gamma}{\alpha}\right)^n \binom{N}{n} =$$

$$= \frac{\sum_{n=0}^{\infty} n \left(\frac{\gamma}{\alpha}\right)^n \binom{N}{n}}{\sum_{n=0}^{\infty} \left(\frac{\gamma}{\alpha}\right)^n \binom{N}{n}} = -\gamma \cdot \frac{\partial}{\partial \gamma} \ln p_0^s$$

$$= +\gamma \cdot N \cdot \frac{\partial}{\partial \gamma} \ln\left(1 + \frac{\gamma}{\alpha}\right) = \frac{\gamma \alpha}{\alpha + \gamma} N \cdot \frac{1}{\alpha} = N \cdot \frac{\gamma}{\alpha + \gamma}$$

and

$$\frac{V_1}{V_2} = \frac{\langle n \rangle}{N - \langle n \rangle} = \frac{\gamma}{\alpha}$$