# Wiener-Khinchine Theorem

$X(t)$ - stationary stochastic process with $\langle X(t) \rangle = 0$

$$X_T(t) = \begin{cases} X(t) & t \in [0,T[ \\ 0 & t \notin [0,T[ \end{cases}$$

Define $A_T(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \, X_T(t) e^{-i\omega t} = \frac{1}{\sqrt{2\pi}} \int_0^T X(t) e^{-i\omega t} dt$

The inverse transform reads $X_T(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A_T(\omega) e^{i\omega t} dt$

① Parseval Identity $\int_{-\infty}^{+\infty} |X_T(t)|^2 dt = \int_{-\infty}^{+\infty} |A_T(\omega)|^2 d\omega$

$$\Rightarrow \int_0^T |X_T(t)|^2 dt = T \langle |X_T|^2 \rangle = \int_{-\infty}^{+\infty} |A_T(\omega)|^2 d\omega$$

So we find that $\langle |X_T|^2 \rangle = \frac{1}{T} \int_{-\infty}^{+\infty} |A_T(\omega)|^2 d\omega$

Thus we find that

$$\lim_{T\to\infty} \langle |X_T|^2 \rangle = \langle |X|^2 \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{+\infty} |A_T(\omega)|^2 d\omega$$

$$:= \int_{-\infty}^{\infty} S(\omega) d\omega$$

So that

$$S(\omega) = \lim_{T\to\infty} \frac{1}{T} \langle |A_T(\omega)|^2 \rangle$$

Let us examine $\quad K(t,t') = \langle X(t)X(t')\rangle$

$$S(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle |A_T(\omega)|^2\right\rangle = \lim_{T\to\infty}\frac{1}{T}\left\langle A_T(\omega)A_T^*(\omega)\right\rangle =$$

$$= \lim_{T\to\infty} \frac{1}{2\pi T}\left\langle \int_0^T X(t)e^{-i\omega t}dt \cdot \int_0^T X(t')e^{i\omega t'}dt'\right\rangle =$$

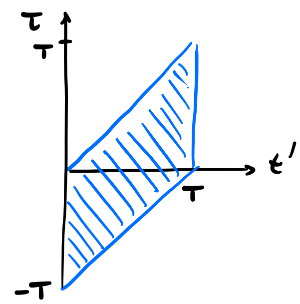
$$= \lim_{T\to\infty} \frac{1}{2\pi}\frac{1}{T}\left\langle \int_0^T\int_0^T dt\,dt'\, X(t)X(t')e^{i\omega(t-t')}\right\rangle =$$

$$= \lim_{T\to\infty} \frac{1}{2\pi}\frac{1}{T}\int_0^T\int_0^T dt\,dt'\, \underbrace{\langle X(t)X(t')\rangle}_{K(t,t')} e^{i\omega(t-t')} =$$

$$= \lim_{T\to\infty} \frac{1}{2\pi T}\int_0^T\int_0^T dt\,dt'\, e^{i\omega(t-t')}K(|t-t'|)$$

$X$ is stationary

$K(t,t') = K(|t-t'|)$




Now change variables to $t$ and $\tau = t-t'$.

$$S(\omega) = \frac{1}{2\pi}\frac{1}{T}\left(\int_{-T}^{0}d\tau\int_0^{T+\tau}dt + \int_0^T d\tau\int_\tau^T dt\right)e^{i\omega\tau}K(\tau) =$$

$$= \frac{1}{2\pi}\frac{1}{T}\left(\int_{-T}^{0}d\tau\,(\tau+T) + \int_0^T d\tau\,(T-\tau)\right)e^{i\omega\tau}K(\tau) =$$

$$= \frac{1}{2\pi}\left(\int_{-T}^{0}d\tau + \int_0^T d\tau\right)e^{i\omega\tau}K(\tau) = \left\{K(\tau)=K(-\tau)\right\}$$

$$= \frac{1}{2\pi}\int_0^T d\tau\left(e^{i\omega\tau}+e^{-i\omega\tau}\right)K(\tau) = \frac{1}{\pi}\int_0^T K(\tau)\cos\omega\tau.$$