Problem 1

(a) $$\frac{\partial P(\varphi,t)}{\partial t} = D_r \frac{\partial^2 P}{\partial \varphi^2}$$

$$P(\varphi,t) = F(\varphi)T(t)$$

$$F(\varphi)T'(t) = D_r F''(\varphi)T(t)$$

$$\frac{1}{D_r}\frac{T'(t)}{T(t)} = \frac{F''(\varphi)}{F(\varphi)} = -m^2$$

$$\left.\begin{array}{l} T_m(t) = T_0 e^{-m^2 D_r t} \\ F_m(\varphi) = C_m \cos m\varphi + S_m \sin m\varphi \end{array}\right\} P = \sum_{m=0}^{\infty} T_m(t) F_m(\varphi)$$

$$P(\varphi,0) = \delta(\varphi-\varphi_0) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty} \cos m(\varphi-\varphi_0)$$

Therefore

$$F_0(\varphi) = \frac{1}{2\pi}$$

$$F_m(\varphi) = \frac{1}{\pi}\cos m(\varphi-\varphi_0)$$

Together:

$$P(\varphi,t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty} e^{-m^2 D_r t} \cos m(\varphi-\varphi_0) \equiv P(\varphi,t|\varphi_0)$$

Long times $\quad P(\varphi,t) \to \frac{1}{2\pi}\quad$ uniform distribution on the circle

(b) External torque $V(\varphi) = -\mu E \cos\varphi$

Equilibrium distribution becomes $P^e \sim e^{-\beta V}$

Probability current

$$\frac{\partial P}{\partial t} = -\frac{\partial j}{\partial \varphi}$$

$$\frac{\partial P^e}{\partial t} = 0 \quad \Rightarrow \quad j(\varphi) = 0$$

(not constant, since this is equilibrium)

$$j = \underset{\substack{\uparrow \\ \text{dissipation}}}{j_r} + \underset{\substack{\uparrow \\ \text{external torque}}}{j_{ext}}$$

$$j_r = -D_r \frac{\partial P}{\partial \varphi} \underset{\substack{\uparrow \\ \text{in equilibrium}}}{=} -D_r \frac{\partial}{\partial \varphi}\left(\frac{e^{-\beta V}}{Z}\right) = -D_r P \cdot -\beta \frac{\partial V}{\partial \varphi} =$$

$$= + D_r P \frac{\partial \beta V}{\partial \varphi}$$

Thus $\quad j_{ext} = -D_r P \dfrac{\partial(\beta V)}{\partial \varphi}$

$$j = -\left[D_r \frac{\partial P}{\partial \varphi} + D_r P \frac{\partial \beta V}{\partial \varphi}\right]$$

The resulting Smoluchowski equation:

$$\frac{\partial P}{\partial t} = D_r \frac{\partial}{\partial \varphi}\left(\frac{\partial P}{\partial \varphi} + P \frac{\partial \beta V}{\partial \varphi}\right)$$

For $V = -\mu E \cos\varphi$ we get

$$\frac{\partial P}{\partial t} = D_r \frac{\partial}{\partial \varphi}\left(\frac{\partial P}{\partial \varphi} + \frac{P}{kT} \mu E \sin\varphi\right)$$

$$= D_r \frac{\partial}{\partial \varphi}\left(\frac{\partial P}{\partial \varphi} + P \xi \sin\varphi\right)$$

(c) Debye's limit

$$\xi = \xi(t) = \xi_0 e^{-i\omega t}$$

$$\frac{\partial P}{\partial t} = D_r \frac{\partial}{\partial \varphi}\left(\frac{\partial P}{\partial \varphi} + \xi_0 e^{-i\omega t} \sin\varphi\right)$$

$$\frac{\partial P}{\partial t} - D_r \frac{\partial^2 P}{\partial \varphi^2} = D_r \xi_0 e^{-i\omega t} \cos\varphi \, P$$

If now $P = P_0 + \xi_0 P_1$ with $P_0 = (2\pi)^{-1}$

$$\xi_0 \frac{\partial P_1}{\partial t} - D_r \frac{\partial^2 P_1}{\partial \varphi^2}\xi_0 = D_r \xi_0 P_0 e^{-i\omega t} \cos\varphi$$

$$\Rightarrow \quad \frac{\partial P_1}{\partial t} - D_r \frac{\partial^2 P_1}{\partial \varphi^2} = D_r \frac{e^{-i\omega t}\cos\varphi}{2\pi}$$

This can be solved (e.g. by separation of variables, + homogeneous / inhomogeneous eq.). The solution periodic in φ is indeed

$$P_1(\varphi, t) = \frac{\cos\varphi}{2\pi(1 - i\omega\tau)} e^{-i\omega t} \qquad \tau = \frac{1}{D_r}$$

From this, the response function $\chi(\omega)$ can be calculated.