This is a draft version of the lecture notes. We aim to keep improving it but at the current stage it is most likely far from perfect. Please contact us if you notice any typos, errors, subtle points, or if you have any questions or suggestions for improvements.

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3 Simple examples of random processes

[?]

4 Random walks

some introduction

4.1 Random walk in 1d

We consider a random walk over a 1d lattice. The time is discrete, the lattice is infinite. We denote by $M \in \mathbb{Z}$ the position of the random walker. With $p_M(N)$ we denote the probability distribution of the position after N steps. We assume that, at each step, the random walker can jump to the left or to the right with equal probabilities. Then

$$p_M(N+1) = \frac{1}{2}p_{M-1}(N) + \frac{1}{2}p_{M+1}(N), \tag{4.1}$$

with some initial condition. We choose it to be $p_M(0) = \delta_{M,0}$. The solution is then

$$p_M(N) = \begin{cases} \frac{1}{2^N}\binom{N}{(N+M)/2}, & N, M \text{ both odd or even,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

To compute this probability we should take the ratio of all the paths starting at 0 and reaching M in N steps, to all the possible paths consisting of N steps. The latter number is 2^N. To compute the former number it is convenient to count number L of steps to the left and number of steps to the right R. We have

$$R + L = N, \quad R - L = M, \tag{4.3}$$

To count number of correct paths is now equivalent to a number of ways in which we can choose R elements from a set of N elements. This number is $\binom{N}{R}$. Writing $R = (N+M)/2$ we obtain the formula. If we start at $M = 0$, the random walk in N steps cannot reach further than $\pm N$. Therefore we should have $p_N(M) = 0$ for $|M| > N$. This is automatically taken care of by the binomial coefficient for which

$$\binom{n}{k} = 0, \qquad \text{for } k > n, k < 0. \tag{4.4}$$

Finally, we observe that the probability distribution has symmetry $p_{-M}(N) = p_M(N)$.

We will now extract features of the probability distribution by computing its moments. The moments of the distribution can be computed with the help of a generating function. Namely

$$f(N,z) = \sum_{M=-N}^{M=N} z^M p_M(N). \tag{4.5}$$

Indeed

$$\langle M\rangle = \sum_{M=-N}^{N} M p_M(N) = \frac{\partial f(N,z)}{\partial z}|_{z=1}, \tag{4.6}$$

$$\langle M^2\rangle = \sum_{M=-N}^{N} M^2 p_M(N) = \frac{\partial^2 f(N,z)}{\partial z^2}|_{z=1} + \langle M\rangle. \tag{4.7}$$

To compute the generating function we use the binomial theorem

$$f(N,z) = \frac{1}{2^N}\sum_{M=-N}^{N} z^M \binom{N}{(N+M)/2} = \frac{z^{-N}}{2^N}\sum_{R=0}^{N} z^{2R}\binom{N}{R} \tag{4.8}$$

$$= \frac{z^{-N}}{2^N}(1+z^2)^N = \frac{(z^{-1}+z)^N}{2^N}. \tag{4.9}$$

We then find

$$f(N,z) = \frac{(z^{-1}+z)^N}{2^N}. \tag{4.10}$$

The first two moments are now readily computable with the result

$$\langle M\rangle = 0, \qquad \langle M^2\rangle = N. \tag{4.11}$$

If the initial condition is $p_M(0) = \delta_{M,M_0}$ then the solution is modified to $p_{M-M_0}(N)$.

The equation is linear and therefore more complicated initial conditions can be easily implemented too. For example, solution $q_M(N)$ to a problem with initial condition $q_M(0) = \frac{1}{2}\delta_{M,1} + \frac{1}{2}\delta_{M,10}$ is

$$q_M(N) = \frac{1}{2}p_{M-1}(N) + \frac{1}{2}p_{M-10}(N). \tag{4.12}$$

We consider now a random walk with a boundary. First we consider an ideally reflective boundary at lattice sites M_0 The presence of the boundary can be by a *mirror image method*. Namely the probability distribution is

$$q_M(N) = \begin{cases} p_M(N) + p_{2M_0-M}(N), & M < M_0, \\ p_{M_0}(N), & M = M_0, \end{cases} \tag{4.13}$$

where $p_M(N)$ is the solution of the infinite lattice problem. The presence of the boundary can be also taken into account by modifying the equations

$$q_M(N+1) = \frac{1}{2}q_{M-1}(N) + \frac{1}{2}q_{M+1}(N), \qquad M < M_0, \tag{4.14}$$

$$q_{M_0}(N+1) = q_{M_0-1}(N). \tag{4.15}$$

We consider now an absorbing boundary at M_0. We can find the solution again by the mirror image method. It reads

$$q_M(N) = \begin{cases} p_M(N) - p_{2M_0-M}(N), & M < M_0, \\ 0, \quad M = M_0. \end{cases} \tag{4.16}$$

The corresponding equations are

$$q_M(N+1) = \frac{1}{2}q_{M-1}(N) + \frac{1}{2}q_{M+1}(N), \qquad M < M_0, \tag{4.17}$$

$$q_{M_0}(N+1) = 0. \tag{4.18}$$

We note that in this case the total probability is not conserved.

4.2 Asymptotic probability distribution

What's the distribution after many steps? To answer this question we can consider large N limit of $p_N(M)$. Since variance $\langle M^2 \rangle$ goes like N, the typical distance from the origin, after N steps is $N^{1/2}$. This implies that for most walks $M/N \ll 1$.

To compute $p_M(N)$ for large N we will use the Stirling approximation for the factorial

$$N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N, \tag{4.19}$$

and use variable $z = M/N$. For the binomial coefficient we then have

$$\begin{aligned} \binom{N}{(N+M)/2} &= \frac{N!}{[(N+M)/2]![(N-M)2]!} = \frac{N!}{[N/2(1+z)]![N/2(1-z)]!} \\ &\approx \frac{2\times 2^N}{\sqrt{2\pi N}}(1-z^2)^{-1/2}(1+z)^{-N/2(1+z)}(1-z)^{-N/2(1-z)} \end{aligned} \tag{4.20}$$

We now use that $z = M/N \ll 1$. This allows to simplify $\sqrt{1-z^2} \approx 1$ and

$$\begin{aligned} (1\pm z)^{-N/2(1\pm z)} &= \exp\left((-N(1\pm z))\log(1\pm z)\right) \\ &\approx \exp\left(-N(1\pm z)(\pm z - \frac{z^2}{2} + \mathcal{O}(z^3))\right) \\ &= \exp\left(-N\left((\pm z + \frac{z^2}{2}\right)\right), \end{aligned} \tag{4.21}$$

which gives

$$(1+z)^{-N/2(1+z)}(1-z)^{-N/2(1-z)} \approx \exp\left(-\frac{Nz^2}{2}\right). \tag{4.22}$$

Replacing now z with M/N we find the following expression for the asymptotic expansion of the binomial coefficient

$$\binom{N}{(N+M)/2} \approx \frac{2\times 2^N}{\sqrt{2\pi N}}\exp\left(-\frac{M^2}{2N}\right). \tag{4.23}$$

For the probability distribution we then get

$$p_M(N) = \frac{2}{\sqrt{2\pi N}} \exp\left(-\frac{M^2}{2N}\right). \tag{4.24}$$

We find the Gaussian distribution of a mean 0 and with variance N. The factor 2 in the numerator of the prefactor takes into account that for given N only every second value of M is possible.

4.3 Random walk in 3d

We consider now a random walk in a $3d$ space. We will assume that space is continuous however we will keep time discrete. The characteristic unit of time we denote τ. We introduce function $g(\boldsymbol{a})$ which is a probability distribution for a jump by a vector $\boldsymbol{a}$. For the 1d walk of the previous section we had $g(\boldsymbol{a}) = (\delta(\boldsymbol{a} - \hat{x}) + \delta(\boldsymbol{a} + \hat{x})/2$. Now, $g(\boldsymbol{a})$ takes non-zero values for jumps of different lengths and different directions. We will later put some assumption on its form but for now we keep it generic. Given this setup, we can write probability distribution $p(\boldsymbol{r}, t+\tau)$ that the particle is at position $\boldsymbol{r}$ at time $t+\tau$ as a result of the particle jumping from its position $\boldsymbol{r} - \boldsymbol{a}$ at time t by $\boldsymbol{a}$. Namely

$$p(\boldsymbol{r}, t+\tau) = \int p(\boldsymbol{r} - \boldsymbol{a}, t) g(\boldsymbol{a}) \mathrm{d}^3 a. \tag{4.25}$$

We will now assume that a length of a single jump $|\boldsymbol{a}|$ is small compared with a typical distance at which we observe the dynamics. This motivates to expand $p(\boldsymbol{r} - \boldsymbol{a})$ around $\boldsymbol{a}$. We find

$$p(\boldsymbol{r} - \boldsymbol{a}, t) = p(\boldsymbol{r}, t) - \sum_{i=1}^{3} \frac{\partial p(\boldsymbol{r}, t)}{\partial x_i} a_i + \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 p(\boldsymbol{r}, t)}{\partial x_i \partial x_j} a_i a_j + \mathcal{O}(|\boldsymbol{a}|^3). \tag{4.26}$$

If we now assume that $g(\boldsymbol{a})$ is isotropic, namely $g(\boldsymbol{a}) = g(|\boldsymbol{a}|)$, then the second term of the expansion does not contribute to the integral and we find

$$p(\boldsymbol{r}, t+\tau) - p(\boldsymbol{r}, t) = \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 p(\boldsymbol{r}, t)}{\partial x_i \partial x_j} \int a_i a_j g(|\boldsymbol{a}|) \mathrm{d}^3 a. \tag{4.27}$$

Because of the isotropy of $g(\boldsymbol{a})$ only diagonal $i = j$ part of this expression survives the integration. Furthermore, let us introduce the following notation

$$\langle \Delta^2 \rangle = \int \left(a_x^2 + a_y^2 + a_z^2\right) g(|\boldsymbol{a}|) \mathrm{d}^3 a = 3 \int a_x^2 g(|\boldsymbol{a}|) \mathrm{d}^3 a, \tag{4.28}$$

for the second moment of the jump probability distribution. In the second step we used again the isotropy assumption. With this, the equation becomes

$$p(\boldsymbol{r}, t+\tau) - p(\boldsymbol{r}, t) = \frac{\langle \Delta^2 \rangle}{6} \sum_{i=1}^{3} \frac{\partial^2 p(\boldsymbol{r}, t)}{\partial x_i^2}. \tag{4.29}$$

We can assume that we observe the system at time scales much larger than τ. This entitles the expansion of the right hand side of the equation. The final result can be written as

$$\frac{\partial p}{\partial t} = D\nabla^2 \rho, \qquad D = \frac{\langle \Delta^2 \rangle}{6\tau}, \tag{4.30}$$

which is the diffusion equation in $3d$. Note how the diffusion constant relates to microscopic parameters of the system, namely the characteristic time and the variance of the transition amplitude.

We can solve this equation using the Fourier transform. We define

$$p(\boldsymbol{k}) = \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} p(\boldsymbol{x}) \mathrm{d}^3 x, \tag{4.31}$$

and the diffusion equation becomes

$$\frac{\partial p(\boldsymbol{k})}{\partial t} = -Dk^2 p(\boldsymbol{k}). \tag{4.32}$$

For the initial condition $p(\boldsymbol{x}, 0) = \delta(\boldsymbol{x})$ which translates into $p(\boldsymbol{k}, 0) = 1$ in the momentum space, the solution is

$$p(\boldsymbol{k}, t) = e^{-D\boldsymbol{k}^2 t}. \tag{4.33}$$

Finally, we want to transfer back to the real space. Using the inverse Fourier transform we have

$$p(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\boldsymbol{x}\cdot\boldsymbol{x}} p(\boldsymbol{k}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\boldsymbol{x}\cdot\boldsymbol{x} - D\boldsymbol{k}^2 t}. \tag{4.34}$$

To compute the remaining integral we first "complete the square" by writing

$$p(\boldsymbol{x}, t) = e^{-Dtk_0^2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-Dt(\boldsymbol{k} - i\boldsymbol{k}_0)^2}, \qquad k_0 = \frac{\boldsymbol{x}}{2Dt}. \tag{4.35}$$

The integral can be shown to be independent of $\boldsymbol{k}_0$ (as long as it's real). Therefore we can set $k_0 = 0$ and the integral becomes then an ordinary Gaussian integral in $3d$. Using that

$$\int \frac{\mathrm{d}k}{2\pi} e^{-Dtk^2} = \frac{1}{\sqrt{4\pi Dt}}, \tag{4.36}$$

and substiuting for $\boldsymbol{k}_0$ we find

$$\rho(\boldsymbol{x}, t) = \left(\frac{1}{4\pi Dt}\right)^{3/2} e^{-\frac{\boldsymbol{x}^2}{4Dt}}. \tag{4.37}$$