Tensors vs pseudotensors

In physics, the nature of a quantity is defined by how it transforms under a change of coordinates. Consider a coordinate transformation defined by an orthogonal matrix $\boldsymbol{R}$. For normal rotations, the determinant is $\det(\boldsymbol{R}) = +1$. Summation convention is assumed throughout.

However, a **parity transformation** (or spatial inversion) reflects the coordinate axes through the origin: $(x, y, z) \to (-x, -y, -z)$. For this transformation, $\boldsymbol{R} = -\boldsymbol{I}$ (the negative identity matrix), and therefore $\det(\boldsymbol{R}) = -1$. The distinction between "true" quantities and "pseudo" quantities is entirely based on how they behave under orthogonal transformations (where $\det(\boldsymbol{R}) = \pm 1$).

(a) A **true scalar** (or purely a scalar) is a quantity that remains completely unchanged under any orthogonal coordinate transformation, including spatial inversion.

$$S' = S \tag{1}$$

Examples: Mass, time, temperature, kinetic energy, electric charge.

(b) A **pseudoscalar** remains unchanged under rotations but *flips its sign* under a parity transformation.

$$P' = \det(\boldsymbol{R})P \tag{2}$$

Examples: Helicity, and the scalar triple product $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$, $\boldsymbol{B} \cdot \boldsymbol{E}$.

(c) A **true vector** (polar vector) transforms exactly like the position vector $\boldsymbol{r}$. Under a parity transformation, all of its components change sign.

$$V_i' = R_{ij} V_j \tag{3}$$

Examples: Position ($\boldsymbol{r}$), velocity ($\boldsymbol{v}$), acceleration ($\boldsymbol{a}$), force ($\boldsymbol{F}$), electric field ($\boldsymbol{E}$).

(d) A **pseudovector** (axial vector) transforms like a vector under pure rotations, but it gains an extra negative sign under spatial inversion (meaning its components do *not* flip sign when the axes do).

$$A_i' = \det(\boldsymbol{R}) R_{ij} A_j \tag{4}$$

Examples: Angular velocity ($\boldsymbol{\omega}$), angular momentum ($\boldsymbol{L}$), torque ($\boldsymbol{\tau}$), magnetic field ($\boldsymbol{B}$), and vorticity ($\nabla \times \boldsymbol{v}$).

We can generalize these rules to tensors of any rank n. A **true tensor** transforms as:

$$T'_{i_1 i_2 \dots i_n} = R_{i_1 j_1} R_{i_2 j_2} \dots R_{i_n j_n} T_{j_1 j_2 \dots j_n} \tag{5}$$

A **pseudotensor** gains a factor of the determinant:

$$\tilde{T}'_{i_1 i_2 \dots i_n} = \det(\boldsymbol{R}) \left(R_{i_1 j_1} R_{i_2 j_2} \dots R_{i_n j_n} \tilde{T}_{j_1 j_2 \dots j_n} \right) \tag{6}$$

The fundamental reason pseudovectors appear so frequently in 3D physics is due to the cross product. The cross product of two true vectors, $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$, can be written in index notation using the Levi-Civita symbol ϵ_{ijk}:

$$C_i = \epsilon_{ijk} A_j B_k \tag{7}$$

Because the Levi-Civita symbol itself is a **pseudotensor** of rank 3, contracting it with two true vectors produces a pseudovector. Contracting it with one true vector and one pseudovector (like in the Lorentz force law $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$) results in a true vector!