Problem Set 5 – Microhydrodynamics & fluctuations

(Useful relationships) The derivatives of the Oseen tensor and the degenerate quadrupole:

$$\begin{aligned}
\mathcal{G}_{ij} &= \frac{1}{r}\,\delta_{ij} + \frac{1}{r^3}\,x_i x_j, \qquad r = |x| \\
\partial_k \mathcal{G}_{ij} &= \frac{1}{r^3}\,(-\delta_{ij}x_k + \delta_{jk}x_i + \delta_{ik}x_j) - \frac{3}{r^5}\,x_i x_j x_k, \\
\nabla^2 \mathcal{G}_{ij} &= \frac{2}{r^3}\,\delta_{ij} - \frac{6}{r^5}\,x_i x_j, \\
\nabla^2(\partial_k \mathcal{G}_{ij}) &= -\frac{6}{r^5}\,(\delta_{ij}x_k + \delta_{jk}x_i + \delta_{ik}x_j) + \frac{30}{r^7}\,x_i x_j x_k.
\end{aligned}$$

Problem 1: A rotating sphere Show that the flow field of a sphere of radius a rotating with an angular velocity $\boldsymbol{\Omega}$ in the ambient flow $\boldsymbol{\omega}^\infty \times \boldsymbol{x}$ can be represented with just a rotlet. What is the magnitude of the torque exerted by the fluid on the sphere? What is the rotational velocity of a freely suspended sphere in ambient flow?

Problem 2: A sphere in a straining flow Fixing a sphere of radius a in a straining flow $\boldsymbol{E}^\infty$ does not require any force, nor torque. What is the hydrodynamic signature of its presence, i.e. how does it disturb the flow?

(a) We would like to find a singularity representation that will be a solution for this case. To this end, define the disturbance flow and write down the boundary conditions it must satisfy.

(b) Which of the singularities (partially) listed above – Stokeslet, symmetric Stokes dipole, degenerate quadrupole (source dipole), degenerate octupole (source quadrupole), might produce the desired disturbance flow? Find the suitable combination.

(c) Find the expression for the resulting disturbance flow $\boldsymbol{v}^D$ as a function of the straining flow strength $\boldsymbol{E}^\infty$. Show that the dominant term is a stresslet of strength

$$\boldsymbol{S} = \frac{20}{3}\pi\eta_s a^3 \boldsymbol{E}^\infty.$$

Problem 3: Stresslets and suspension stress A dilute suspension of rigid spheres of radius a is placed in an imposed linear flow with rate-of-strain tensor $\boldsymbol{E}^\infty$. From Problem 2 above, we found the hydrodynamic stresslet for a single sphere $\boldsymbol{S} = (20\pi/3)\eta_s a^3 \boldsymbol{E}^\infty$. Let the particle number density be n and the particle volume fraction be $\phi = 4\pi a^3 n/3$.

(a) Show that the particle contribution to the bulk stress is of the form

$$\langle \boldsymbol{\Sigma}^p \rangle = n\boldsymbol{S},$$

and rewrite it in terms of ϕ and $\boldsymbol{E}^\infty$.

(b) Define the effective viscosity by comparing

$$\langle \boldsymbol{\Sigma} \rangle = -\langle p \rangle \boldsymbol{I} + 2\eta_{\text{eff}} \boldsymbol{E}^\infty$$

with the sum of the solvent stress and the particle stress above. Recover Einstein's dilute-limit correction (1906) as

$$\eta_{\text{eff}} = \eta_s \left(1 + \frac{5}{2}\phi\right).$$

Explain why the stresslet, rather than the Stokeslet, is the relevant single-particle quantity here.

Problem 4: Force on an arbitrary particle in an ambient flow Consider a rigid particle of arbitrary shape, fixed in an ambient flow $\boldsymbol{v}^\infty(\boldsymbol{x})$. We shall derive Brenner's result for the drag on the particle [H. Brenner, *Chem. Eng. Sci.* **19**, 703-727 (1964)]. Suppose that $\boldsymbol{v}'$ is the solution for a particle translating with steady velocity $\boldsymbol{U}$ and $\boldsymbol{\sigma}'$ is the associated stress field. Denote the disturbance velocity and stress fields for the solution with the arbitrary ambient flow by $\boldsymbol{v}^D$ and $\boldsymbol{\sigma}^D$.

(a) Apply the Lorentz reciprocal theorem to the disturbance flow $(\boldsymbol{v}^D, \boldsymbol{\sigma}^D)$ and the auxiliary flow $(\boldsymbol{v}', \boldsymbol{\sigma}')$. Show that

$$\boldsymbol{U} \cdot \oint_{S_p} \boldsymbol{\sigma}^D \cdot \boldsymbol{n}\, \mathrm{d}S = -\oint_{S_p} \boldsymbol{v}^\infty \cdot (\boldsymbol{\sigma}' \cdot \boldsymbol{n})\, \mathrm{d}S.$$

(b) Show that this implies that the drag on the particle in an arbitrary ambient field satisfying $\nabla \cdot \boldsymbol{\sigma}^\infty = 0$ is directly related to that ambient field by

$$F_i(\boldsymbol{v}^\infty) = \oint_{S_p} (\boldsymbol{\sigma}^D \cdot \boldsymbol{n})_i \mathrm{d}S = -\oint_{S_p} w_{ij} v_j^\infty \mathrm{d}S,$$

where $U_i w_{ij} = (\boldsymbol{\sigma}' \cdot \boldsymbol{n})_j$ is the j-th component of the surface traction from the problem of the translating particle. This shows that given the surface tractions for the translation problem, one can compute directly the hydrodynamic drag for the same particle in an arbitrary field.

(c) Apply the result to a sphere. Hence show that a freely moving (force-free) sphere in an arbitrary flow will have the velocity

$$\boldsymbol{U} = \frac{1}{4\pi a^2} \int_{r=a} \boldsymbol{v}^\infty(\boldsymbol{x}) \mathrm{d}S.$$

which can be interpreted as the average of the ambient flow field over the sphere surface.

Hint: You may use the classical Stokes solution for a translating sphere in which $\boldsymbol{\sigma}' \cdot \boldsymbol{n} = -3\eta_s \boldsymbol{U}/2a$ on the sphere surface.

Solution sketches

Problem 1: A rotating sphere Let the sphere rotate with angular velocity $\boldsymbol{\Omega}$ in the ambient solid-body rotation

$$\boldsymbol{v}^\infty(\boldsymbol{x}) = \boldsymbol{\omega}^\infty \times \boldsymbol{x}.$$

Define the disturbance flow

$$\boldsymbol{v}^D = \boldsymbol{v} - \boldsymbol{v}^\infty.$$

Since both the imposed flow and the particle motion are purely rotational, the disturbance satisfies

$$\boldsymbol{v}^D = (\boldsymbol{\Omega} - \boldsymbol{\omega}^\infty) \times \boldsymbol{x} \qquad \text{on } r = a,$$

and

$$\boldsymbol{v}^D \to \boldsymbol{0} \qquad (r \to \infty).$$

There is no net force, only a torque, so the disturbance must be represented by a *rotlet*. The rotlet velocity field has the form

$$\boldsymbol{v}^D(\boldsymbol{x}) = \frac{a^3}{r^3}\,(\boldsymbol{\Omega} - \boldsymbol{\omega}^\infty) \times \boldsymbol{x}.$$

Indeed, on $r = a$ this gives $(\boldsymbol{\Omega} - \boldsymbol{\omega}^\infty) \times \boldsymbol{x}$, and it decays as r^{-2} at infinity.

Comparing with the standard rotlet representation

$$\boldsymbol{v}^D(\boldsymbol{x}) = \frac{1}{8\pi\eta_s}\,\frac{\boldsymbol{L} \times \boldsymbol{x}}{r^3},$$

we identify the hydrodynamic torque exerted by the fluid on the sphere as

$$\boldsymbol{L}^H = -\,8\pi\eta_s a^3(\boldsymbol{\Omega} - \boldsymbol{\omega}^\infty).$$

Hence the magnitude of the torque is

$$|\boldsymbol{L}^H| = 8\pi\eta_s a^3\,|\boldsymbol{\Omega} - \boldsymbol{\omega}^\infty|.$$

If the sphere is freely rotating, $\boldsymbol{L}^H = \boldsymbol{0}$, so $\boldsymbol{\Omega} = \boldsymbol{\omega}^\infty$.

Problem 2: A sphere in a straining flow Let the imposed flow be

$$\boldsymbol{v}^\infty(\boldsymbol{x}) = \boldsymbol{E}^\infty \cdot \boldsymbol{x}, \qquad E^\infty_{ij} = E^\infty_{ji}, \qquad E^\infty_{ii} = 0.$$

The sphere is fixed at the origin.

(a) Define the disturbance flow

$$\boldsymbol{v}^D = \boldsymbol{v} - \boldsymbol{v}^\infty.$$

Then $\boldsymbol{v}^D$ satisfies the Stokes equations outside the sphere, with

$$\boldsymbol{v}^D = -\boldsymbol{E}^\infty \cdot \boldsymbol{x} \qquad \text{on } r = a,$$

and

$$\boldsymbol{v}^D \to \boldsymbol{0} \qquad (r \to \infty).$$

Since a pure strain exerts neither net force nor net torque on the sphere, the disturbance cannot contain a Stokeslet or a rotlet.

(b) The disturbance must be built from singularities with the same tensorial symmetry as $\boldsymbol{E}^{\infty}$: a *symmetric Stokes dipole* (stresslet) and a higher-order correction that enforces no-slip on the sphere. A suitable ansatz is

$$v_i^D = A\, E_{jk}^{\infty}\, \partial_k \mathcal{G}_{ij} + B\, E_{jk}^{\infty}\, \nabla^2(\partial_k \mathcal{G}_{ij}).$$

Using the identities given on the sheet and $E_{jj}^{\infty} = 0$,

$$E_{jk}^{\infty} \partial_k \mathcal{G}_{ij} = -\frac{3x_i}{r^5}\, x_j E_{jk}^{\infty} x_k,$$

and

$$E_{jk}^{\infty} \nabla^2(\partial_k \mathcal{G}_{ij}) = -\frac{12}{r^5}(E^{\infty} \cdot x)_i + \frac{30x_i}{r^7}\, x_j E_{jk}^{\infty} x_k.$$

Matching the boundary condition $\boldsymbol{v}^D = -\boldsymbol{E}^{\infty} \cdot \boldsymbol{x}$ on $r = a$ gives

$$A = \frac{5a^3}{6}, \qquad B = \frac{a^5}{12}.$$

So the required singularity combination is a symmetric Stokes dipole plus a degenerate octupole (source quadrupole):

$$v_i^D = \frac{5a^3}{6}\, E_{jk}^{\infty} \partial_k \mathcal{G}_{ij} + \frac{a^5}{12}\, E_{jk}^{\infty} \nabla^2(\partial_k \mathcal{G}_{ij}).$$

(c) Substituting the contracted identities above gives

$$v_i^D = -\frac{a^5}{r^5}(E^{\infty} \cdot x)_i - \frac{5a^3}{2}\, \frac{r^2 - a^2}{r^7}\, x_i\, (x \cdot E^{\infty} \cdot x).$$

Equivalently,

$$\boldsymbol{v}^D = -\frac{a^5}{r^5}\, \boldsymbol{E}^{\infty} \cdot \boldsymbol{x} + \frac{5a^3}{2}\left(\frac{a^2}{r^2} - 1\right) \frac{\boldsymbol{x}\, (\boldsymbol{x} \cdot \boldsymbol{E}^{\infty} \cdot \boldsymbol{x})}{r^5}.$$

One may check that this satisfies $\boldsymbol{v}^D = -\boldsymbol{E}^{\infty} \cdot \boldsymbol{x}$ on $r = a$ and $\boldsymbol{v}^D \to 0$ as $r \to \infty$.

For $r \gg a$, the leading term is

$$v_i^D \sim \frac{5a^3}{6}\, E_{jk}^{\infty} \partial_k \mathcal{G}_{ij},$$

which is precisely a stresslet field. Comparing with the standard stresslet form

$$v_i^S = \frac{1}{8\pi\eta_s}\, S_{jk}\, \partial_k \mathcal{G}_{ij},$$

we identify

$$\boxed{\boldsymbol{S} = \frac{20\pi}{3}\eta_s a^3 \boldsymbol{E}^{\infty}.}$$

Problem 3: Stresslets and suspension stress For a dilute suspension, particle interactions are negligible to leading order, so the particle contribution to the bulk stress is the number of particles per unit volume times the single-particle stresslet.

(a) If there are N particles in a volume V, then

$$\langle \boldsymbol{\Sigma}^p \rangle = \frac{1}{V} \sum_{\alpha=1}^{N} \boldsymbol{S}^{(\alpha)}.$$

In the dilute limit, every sphere has the same stresslet $\boldsymbol{S}$, so

$$\langle \boldsymbol{\Sigma}^p \rangle = \frac{N}{V} \boldsymbol{S} = n\boldsymbol{S}.$$

Using

$$\boldsymbol{S} = \frac{20\pi}{3} \eta_s a^3 \boldsymbol{E}^\infty, \qquad \phi = \frac{4\pi a^3}{3} n,$$

we find

$$\langle \boldsymbol{\Sigma}^p \rangle = n \frac{20\pi}{3} \eta_s a^3 \boldsymbol{E}^\infty = 5\eta_s \phi \, \boldsymbol{E}^\infty.$$

(b) The total bulk stress is the solvent stress plus the particle stress:

$$\langle \boldsymbol{\Sigma} \rangle = -\langle p \rangle \boldsymbol{I} + 2\eta_s \boldsymbol{E}^\infty + 5\eta_s \phi \, \boldsymbol{E}^\infty.$$

Factor out $2\boldsymbol{E}^\infty$:

$$\langle \boldsymbol{\Sigma} \rangle = -\langle p \rangle \boldsymbol{I} + 2\eta_s \left(1 + \frac{5}{2}\phi\right) \boldsymbol{E}^\infty.$$

Comparing with

$$\langle \boldsymbol{\Sigma} \rangle = -\langle p \rangle \boldsymbol{I} + 2\eta_{\text{eff}} \boldsymbol{E}^\infty,$$

gives Einstein's dilute-limit formula

$$\boxed{\eta_{\text{eff}} = \eta_s \left(1 + \frac{5}{2}\phi\right).}$$

The stresslet, rather than the Stokeslet, is the relevant single-particle quantity because a neutrally suspended rigid sphere in a linear flow is *force-free* and *torque-free*. Hence there is no Stokeslet contribution. The leading nonzero moment of the surface traction is the symmetric force dipole, i.e. the stresslet, and this is the quantity that enters the bulk suspension stress.

Problem 4: Force on an arbitrary particle in an ambient flow Let the fluid domain exterior to the particle be V_f, with particle surface S_p. Since the particle is fixed in the actual problem, the disturbance field

$$\boldsymbol{v}^D = \boldsymbol{v} - \boldsymbol{v}^\infty$$

satisfies the Stokes equations in V_f, with boundary conditions

$$\boldsymbol{v}^D = -\boldsymbol{v}^\infty \quad \text{on } S_p, \qquad \boldsymbol{v}^D \to \boldsymbol{0} \quad \text{as } |\boldsymbol{x}| \to \infty.$$

In the auxiliary problem, the same particle translates rigidly with velocity $\boldsymbol{U}$, so

$$\boldsymbol{v}' = \boldsymbol{U} \quad \text{on } S_p, \qquad \boldsymbol{v}' \to \boldsymbol{0} \quad \text{as } |\boldsymbol{x}| \to \infty.$$

(a) Apply the Lorentz reciprocal theorem to $(\boldsymbol{v}^D, \boldsymbol{\sigma}^D)$ and $(\boldsymbol{v}', \boldsymbol{\sigma}')$:

$$\int_{\partial V_f} \boldsymbol{v}^D \cdot (\boldsymbol{\sigma}' \cdot \boldsymbol{n}) \, dS = \int_{\partial V_f} \boldsymbol{v}' \cdot (\boldsymbol{\sigma}^D \cdot \boldsymbol{n}) \, dS.$$

The boundary ∂V_f consists of the particle surface S_p and a large sphere at infinity. Since both $\boldsymbol{v}^D$ and $\boldsymbol{v}'$ decay at infinity, the contribution from the outer boundary vanishes, leaving

$$\oint_{S_p} \boldsymbol{v}^D \cdot (\boldsymbol{\sigma}' \cdot \boldsymbol{n}) \, dS = \oint_{S_p} \boldsymbol{v}' \cdot (\boldsymbol{\sigma}^D \cdot \boldsymbol{n}) \, dS.$$

Now use the boundary conditions on S_p:

$$\boldsymbol{v}^D = -\boldsymbol{v}^\infty, \qquad \boldsymbol{v}' = \boldsymbol{U}.$$

Then

$$-\oint_{S_p} \boldsymbol{v}^\infty \cdot (\boldsymbol{\sigma}' \cdot \boldsymbol{n})\, dS = \boldsymbol{U} \cdot \oint_{S_p} \boldsymbol{\sigma}^D \cdot \boldsymbol{n}\, dS.$$

Hence

$$\boldsymbol{U} \cdot \oint_{S_p} \boldsymbol{\sigma}^D \cdot \boldsymbol{n}\, dS = -\oint_{S_p} \boldsymbol{v}^\infty \cdot (\boldsymbol{\sigma}' \cdot \boldsymbol{n})\, dS,$$

as required.

(b) Define the hydrodynamic force on the fixed particle in the ambient flow by

$$F_i(\boldsymbol{v}^\infty) := \oint_{S_p} (\boldsymbol{\sigma}^D \cdot \boldsymbol{n})_i\, dS.$$

Because the auxiliary Stokes problem is linear in the translation velocity $\boldsymbol{U}$, the traction $\boldsymbol{\sigma}' \cdot \boldsymbol{n}$ depends linearly on $\boldsymbol{U}$. Therefore there exists a tensor field $w_{ij}(\boldsymbol{x})$ on the particle surface such that

$$(\boldsymbol{\sigma}' \cdot \boldsymbol{n})_j = U_i w_{ij}.$$

Substitute this into the result of part (a):

$$U_i F_i = -\oint_{S_p} v_j^\infty\, U_i w_{ij}\, dS = U_i \left(-\oint_{S_p} w_{ij} v_j^\infty\, dS \right).$$

Since $\boldsymbol{U}$ is arbitrary, we may equate coefficients of U_i, obtaining

$$F_i(\boldsymbol{v}^\infty) = -\oint_{S_p} w_{ij} v_j^\infty\, dS.$$

Thus the drag in an arbitrary ambient field is determined directly from the surface tractions of the pure translation problem.

(c) For a sphere of radius a, the classical Stokes solution for rigid translation gives

$$\boldsymbol{\sigma}' \cdot \boldsymbol{n} = -\frac{3\eta_s}{2a}\boldsymbol{U} \qquad \text{on } r = a.$$

Hence

$$(\boldsymbol{\sigma}' \cdot \boldsymbol{n})_j = -\frac{3\eta_s}{2a} U_j,$$

so that

$$w_{ij} = -\frac{3\eta_s}{2a}\delta_{ij}.$$

Substituting into the formula from part (b),

$$F_i(\boldsymbol{v}^\infty) = -\oint_{S_p} \left(-\frac{3\eta_s}{2a}\delta_{ij} \right) v_j^\infty\, dS = \frac{3\eta_s}{2a} \oint_{S_p} v_i^\infty\, dS.$$

Now consider a freely moving sphere with translational velocity $\boldsymbol{U}$ in the ambient flow $\boldsymbol{v}^\infty$. In the instantaneous rest frame of the sphere, the relevant ambient field is $\boldsymbol{v}^\infty - \boldsymbol{U}$. Since the sphere is force-free,

$$0 = \frac{3\eta_s}{2a} \oint_{r=a} \left(v_i^\infty - U_i \right) dS.$$

Therefore

$$\oint_{r=a} v_i^\infty \, dS = U_i \oint_{r=a} dS = 4\pi a^2 U_i.$$

Thus

$$U_i = \frac{1}{4\pi a^2} \int_{r=a} v_i^\infty(\boldsymbol{x}) \, dS,$$

or in vector form,

$$\boldsymbol{U} = \frac{1}{4\pi a^2} \int_{r=a} \boldsymbol{v}^\infty(\boldsymbol{x}) \, dS.$$

So a freely moving sphere translates with the surface average of the ambient flow.

Tensors vs pseudotensors

In physics, the nature of a quantity is defined by how it transforms under a change of coordinates. Consider a coordinate transformation defined by an orthogonal matrix $\boldsymbol{R}$. For normal rotations, the determinant is $\det(\boldsymbol{R}) = +1$. Summation convention is assumed throughout.

However, a **parity transformation** (or spatial inversion) reflects the coordinate axes through the origin: $(x, y, z) \to (-x, -y, -z)$. For this transformation, $\boldsymbol{R} = -\boldsymbol{I}$ (the negative identity matrix), and therefore $\det(\boldsymbol{R}) = -1$. The distinction between "true" quantities and "pseudo" quantities is entirely based on how they behave under orthogonal transformations (where $\det(\boldsymbol{R}) = \pm 1$).

(a) A **true scalar** (or purely a scalar) is a quantity that remains completely unchanged under any orthogonal coordinate transformation, including spatial inversion.

$$S' = S \tag{1}$$

Examples: Mass, time, temperature, kinetic energy, electric charge.

(b) A **pseudoscalar** remains unchanged under rotations but *flips its sign* under a parity transformation.

$$P' = \det(\boldsymbol{R}) P \tag{2}$$

Examples: Helicity, and the scalar triple product $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$, $\boldsymbol{B} \cdot \boldsymbol{E}$.

(c) A **true vector** (polar vector) transforms exactly like the position vector $\boldsymbol{r}$. Under a parity transformation, all of its components change sign.

$$V'_i = R_{ij} V_j \tag{3}$$

Examples: Position ($\boldsymbol{r}$), velocity ($\boldsymbol{v}$), acceleration ($\boldsymbol{a}$), force ($\boldsymbol{F}$), electric field ($\boldsymbol{E}$).

(d) A **pseudovector** (axial vector) transforms like a vector under pure rotations, but it gains an extra negative sign under spatial inversion (meaning its components do *not* flip sign when the axes do).

$$A'_i = \det(\boldsymbol{R}) R_{ij} A_j \tag{4}$$

Examples: Angular velocity ($\boldsymbol{\omega}$), angular momentum ($\boldsymbol{L}$), torque ($\boldsymbol{\tau}$), magnetic field ($\boldsymbol{B}$), and vorticity ($\nabla \times \boldsymbol{v}$).

We can generalize these rules to tensors of any rank n. A **true tensor** transforms as:

$$T'_{i_1 i_2 \ldots i_n} = R_{i_1 j_1} R_{i_2 j_2} \ldots R_{i_n j_n} T_{j_1 j_2 \ldots j_n} \tag{5}$$

A **pseudotensor** gains a factor of the determinant:

$$\tilde{T}'_{i_1 i_2 \ldots i_n} = \det(\boldsymbol{R}) \left(R_{i_1 j_1} R_{i_2 j_2} \ldots R_{i_n j_n} \tilde{T}_{j_1 j_2 \ldots j_n} \right) \tag{6}$$

The fundamental reason pseudovectors appear so frequently in 3D physics is due to the cross product. The cross product of two true vectors, $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$, can be written in index notation using the Levi-Civita symbol ϵ_{ijk}:

$$C_i = \epsilon_{ijk} A_j B_k \tag{7}$$

Because the Levi-Civita symbol itself is a **pseudotensor** of rank 3, contracting it with two true vectors produces a pseudovector. Contracting it with one true vector and one pseudovector (like in the Lorentz force law $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$) results in a true vector!