

Problem Set 4 – Microhydrodynamics & fluctuations

Problem 1: stress field of the Oseen tensor

Consider the stress field, denoted by $\mathbf{F} \cdot \boldsymbol{\Sigma}$, of the Green's function $\mathbf{G}$. The stress field of the Oseen tensor is then the triadic $8\pi\eta_s\boldsymbol{\sigma}$. Show that

$$8\pi\eta_s\Sigma_{ijk} = -6\eta_s\frac{r_i r_j r_k}{r^5}.$$

With the singular point being placed at $\boldsymbol{\xi}$, show the following statements:

$$8\pi\eta_s\frac{\partial}{\partial x_k}\Sigma_{ijk}(\mathbf{x} - \boldsymbol{\xi}) = -\frac{\partial}{\partial x_i}\mathcal{P}_j(\mathbf{x} - \boldsymbol{\xi}) + \eta_s\nabla^2\mathbf{G}_{ij} = -8\pi\eta_s\delta_{ij}\delta(\mathbf{x} - \boldsymbol{\xi}),$$

$$\frac{\partial}{\partial x_i}\mathbf{G}_{ij}(\mathbf{x} - \boldsymbol{\xi}) = 0.$$

Problem 2: Representation of flow outside a rigid particle

Consider an ambient Stokes flow (in the absence of particles), $\mathbf{v}^\infty$. A rigid particle immersed in this flow, and moving with a translational velocity $\mathbf{U}$ and rotational velocity $\boldsymbol{\Omega}$, creates a disturbance flow, such that the velocity field in the presence of the particle, $\mathbf{v}$ matches the particle (rigid body) motion, $\mathbf{v}^{RBM}$ on its surface, and the ambient flow far away. Define the disturbance field $\mathbf{v}^D = \mathbf{v} - \mathbf{v}^\infty$.

- (a) Write the integral representation for $\mathbf{v}^D$.
- (b) Now apply the integral representation theorem to the field $\mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r} - \mathbf{v}^\infty$ inside the particle. Mind the orientation of the normal vector!
- (c) Add the representations (for $\mathbf{v}^D$ and $\mathbf{v}^{RBM} - \mathbf{v}^\infty$), and obtain the result that the flow past a rigid particle can be represented with just single layer potentials.

Hints: for a rigid-body motion, show that the surface traction $\boldsymbol{\sigma}^{RBM} \cdot \mathbf{n} = -p_0\mathbf{n}$ and that since $\mathbf{G}$ is solenoidal, the integral of $\mathbf{n} \cdot \mathbf{G}$ over the surface of the particle is identically zero.

Problem 3: Constant single layer density on a circular disk Consider a constant single layer density ψ distributed over a circular disk of radius R . Show by direct integration in cylindrical coordinates that the velocity field generated by this distribution is continuous passing through the disk. Are the tractions also continuous?

Problem 4: Components of the double layer potential

Consider the decomposition of the double layer kernel into the Oseen pressure field and the rate-of-strain field of the Oseen tensor. Show that

$$8\pi\eta_s(\boldsymbol{\Sigma}(\mathbf{x} - \boldsymbol{\xi}) \cdot \mathbf{n}) \cdot \boldsymbol{\varphi} = -(\boldsymbol{\varphi} \cdot \mathbf{n})\mathcal{P}(\mathbf{x} - \boldsymbol{\xi}) + \eta_s(\boldsymbol{\varphi}\mathbf{n} + \mathbf{n}\boldsymbol{\varphi}) : \nabla\mathbf{G}(\mathbf{x} - \boldsymbol{\xi}),$$

where $:$ denotes double contraction.

The first term corresponds to sources and sinks, depending on the sign of the density, $\boldsymbol{\varphi}$, while the second group corresponds to a true bilayer of Stokeslets. Argue that the source/sink term gives a jump in the normal, but continuous tangential velocities, while for the bilayer the opposite occurs. Verify this by considering a local analysis using a constant density on a plane. In that situation, show also that only the tangential component of double layer density, $\boldsymbol{\varphi}^\perp = \boldsymbol{\varphi} - \boldsymbol{\varphi} \cdot \mathbf{n}\mathbf{n}$, contributes to the velocity field of the bilayer distribution.

Hint: The double contraction of two tensors is $(A : B)_{i_1 \dots i_m j_1 \dots j_n} = A_{i_1 \dots i_m p q} B_{p q j_1 \dots j_n}$.