## Problem Set 3 – Microhydrodynamics & fluctuations

### Problem 1: Derivation of the Oseen tensor

Consider the incompressible Stokes equations in $\mathbb{R}^3$ with a point force applied at the origin,

$$-\nabla p + \eta \nabla^2 \boldsymbol{v} + \boldsymbol{F}\,\delta(\boldsymbol{r}) = \boldsymbol{0}, \qquad \nabla \cdot \boldsymbol{v} = 0.$$

The free-space Green's tensor of the Stokes equation is the Oseen tensor $\boldsymbol{G}$. Solve the Stokes equations and find the form of $\boldsymbol{G}$ in an unbounded fluid.[1] Use the Fourier transform to derive the fundamental solution.

(a) Fourier transform the equations and solve for $\hat{\boldsymbol{v}}(\boldsymbol{k})$ and $\hat{p}(\boldsymbol{k})$. Show that $\hat{\boldsymbol{v}}$ can be written using the transverse projector

$$\mathsf{P}_{ij}(\boldsymbol{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}.$$

(b) Show that the inverse transform of $\hat{u}_i(\boldsymbol{k})$ is necessarily of the form

$$G_{ij}(\boldsymbol{r}) = A\,\frac{\delta_{ij}}{r} + B\,\frac{r_i r_j}{r^3},$$

by rotational symmetry, and determine $A$ and $B$ from the Fourier-space expression.

(c) Deduce the corresponding pressure field and verify that for $r \neq 0$ the pair $(\boldsymbol{u}, p)$ satisfies the homogeneous Stokes equations and incompressibility.

(d) In (b), the normalisation constant is calculated directly. Show that, for the Green's tensor in the form $A(\delta_{ij}/r + r_i r_j/r^3)$, as argued in the lecture, the constant $A$ can also be fixed by computing the stress tensor associated with the velocity field, integrating the traction over a sphere $S_R$ centred at the origin, and usin the defining property of the Green function

$$\int_{S_R} \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dS = -\boldsymbol{F},$$

that is, that the total force acting at the origin is $\boldsymbol{F}$.

### Problem 2: Multipole structure beyond the Stokeslet

Let a localised force distribution $f_j(\boldsymbol{y})$ act in a small region around the origin. The corresponding velocity field may be written as

$$u_i(\boldsymbol{x}) = \frac{1}{8\pi\eta} \int G_{ij}(\boldsymbol{x} - \boldsymbol{y}) f_j(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}.$$

Assume $|\boldsymbol{x}|$ is much larger than the size of the forcing region.

(a) Perform a Taylor expansion of $G_{ij}(\boldsymbol{x} - \boldsymbol{y})$ about $\boldsymbol{y} = 0$ up to first order and show that

$$u_i(\boldsymbol{x}) = \frac{1}{8\pi\eta} \left[ G_{ij}(\boldsymbol{x}) F_j - \partial_k G_{ij}(\boldsymbol{x}) D_{jk} + \cdots \right],$$

where

$$F_j = \int f_j(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}, \qquad D_{jk} = \int y_k f_j(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}.$$

Interpret $F_j$ and $D_{jk}$ physically.

[1] There are at least five different ways of systematically deriving the Oseen tensor. Four of them are listed in https://doi.org/10.48550/arXiv.1312.6231 by ML.

(b) Decompose the dipole tensor as

$$D_{jk} = \frac{1}{3}(\mathrm{tr}\, D)\delta_{jk} + \frac{1}{2}(D_{jk} - D_{kj}) + \left[\frac{1}{2}(D_{jk} + D_{kj}) - \frac{1}{3}(\mathrm{tr}\, D)\delta_{jk}\right].$$

Show that the three pieces generate, respectively, a source-dipole-type contribution, a rotlet, and a stresslet. Determine the far-field decay of each term. Then consider a force-free and torque-free object and explain why the stresslet is generically the leading far-field singularity.

(c) Consider two equal and opposite point forces $\pm \boldsymbol{f}$ acting at $\pm\frac{1}{2}\ell\, \boldsymbol{e}$. Compute the dipole tensor explicitly and determine under what condition the far field contains a pure stresslet with no rotlet part.

**Problem 3: Lorentz reciprocity and the structure of friction/mobility relations**

Consider a rigid particle of arbitrary shape in an unbounded Stokes flow. In experiment A, the particle moves with translational and angular velocity $(\boldsymbol{U}^{(A)}, \boldsymbol{\Omega}^{(A)})$ in a quiescent fluid, and the fluid exerts hydrodynamic force and torque $(\boldsymbol{F}^{(A)}, \boldsymbol{T}^{(A)})$ on the particle. In experiment B, the corresponding quantities are $(\boldsymbol{U}^{(B)}, \boldsymbol{\Omega}^{(B)})$ and $(\boldsymbol{F}^{(B)}, \boldsymbol{T}^{(B)})$.

(a) Using the Lorentz reciprocal theorem, prove that

$$\boldsymbol{F}^{(A)} \cdot \boldsymbol{U}^{(B)} + \boldsymbol{T}^{(A)} \cdot \boldsymbol{\Omega}^{(B)} = \boldsymbol{F}^{(B)} \cdot \boldsymbol{U}^{(A)} + \boldsymbol{T}^{(B)} \cdot \boldsymbol{\Omega}^{(A)}.$$

Deduce that the $6 \times 6$ friction matrix

$$\begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{T} \end{pmatrix} = -\begin{pmatrix} \boldsymbol{\zeta}^{tt} & \boldsymbol{\zeta}^{tr} \\ \boldsymbol{\zeta}^{rt} & \boldsymbol{\zeta}^{rr} \end{pmatrix} \begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{\Omega} \end{pmatrix}$$

is symmetric in the sense that

$$\boldsymbol{\zeta}^{tt} = (\boldsymbol{\zeta}^{tt})^\top, \qquad \boldsymbol{\zeta}^{rr} = (\boldsymbol{\zeta}^{rr})^\top, \qquad \boldsymbol{\zeta}^{tr} = (\boldsymbol{\zeta}^{rt})^\top.$$

(b) Assume that the mobility relation exists,

$$\begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{\Omega} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}^{tt} & \boldsymbol{\mu}^{tr} \\ \boldsymbol{\mu}^{rt} & \boldsymbol{\mu}^{rr} \end{pmatrix} \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{T} \end{pmatrix}.$$

Show directly from the reciprocity identity above, without using linear algebra facts about inverses of symmetric matrices, that the mobility matrix satisfies the analogous symmetry relations

$$\boldsymbol{\mu}^{tt} = (\boldsymbol{\mu}^{tt})^\top, \qquad \boldsymbol{\mu}^{rr} = (\boldsymbol{\mu}^{rr})^\top, \qquad \boldsymbol{\mu}^{tr} = (\boldsymbol{\mu}^{rt})^\top.$$

Discuss briefly why these symmetry relations are physically nontrivial for a particle of non-spherical shape.

**Problem 4: Force-, torque-, and stresslet-free? What is transmitted to infinity?** Let $S_1$ and $S_2$ be two smooth, closed surfaces with $S_1$ entirely enclosed by $S_2$, and let the Stokes flow occupy the region between them. Assume that there are no body forces in this region.

(a) Using the divergence theorem and the choice of an auxiliary rigid-body motion in the Lorentz reciprocal theorem, show that the total hydrodynamic force and torque exerted across $S_1$ are transmitted unchanged to $S_2$.

(b) Now choose as auxiliary flow a linear ambient field $\boldsymbol{u}' = \boldsymbol{E} \cdot \boldsymbol{x}$ with constant, symmetric, traceless $\boldsymbol{E}$. Show that the symmetric first moment of traction (the stresslet) is also transmitted from $S_1$ to $S_2$ provided there are no body forces in the region between the two surfaces. Explain why this does *not* imply that a force-free particle has a vanishing far field.