# Problem Set 2 – Microhydrodynamics & fluctuations

**Problem 1: Vanishing strain-rate fields**

Consider a fluid with vanishing strain-rate tensor, i.e. $\mathbf{e} = 0$ on fluid domain $\mathcal{V}$. Show that the corresponding velocity field $\mathbf{v}$ is at most a uniform translation plus a solid-body rotation.

**Problem 2: Lower and upper bounds on energy dissipation**

In the lecture we have derived an upper bound for the rate of energy dissipation $\Phi$ due to viscosity for a Newtonian fluid. This result is encapsulated in the so-called minimum energy dissipation theorem for Stokes flows.

(a) Write down the explicit statement of this theorem, and under which conditions it holds. Denote the upper bound of $\Phi$ by $\Phi^*$, the fluid domain by $\mathcal{V}$, and the bounding surface of the fluid by $\mathcal{S}$.

(b) One can also obtain a lower bound for $\Phi$. Consider an equilibrium stress field that satisfies $\nabla \cdot \boldsymbol{\sigma}' = 0$ and define $2\eta_{\mathrm{s}}\mathbf{e}' = \boldsymbol{\sigma}' + p'\mathbf{I}$, with $p'$ determined such that $e'_{\nu\nu} = 0$. Here $\mathbf{e}'$ is not necessarily associated with a flow field. Prove that

$$2 \oint_{\mathcal{S}} dS\, \mathbf{v} \cdot (\boldsymbol{\sigma}' \cdot \hat{\mathbf{n}}) - \Phi' \leq \Phi, \tag{1}$$

where $\Phi' = \int_{\mathcal{V}} dV\, \boldsymbol{\sigma}' : \mathbf{e}'$. In which direction does the unit normal $\hat{\mathbf{n}}$ point?

(c) We apply the bound derived in (b) to a particle suspended in an unbounded viscous fluid. We take a rigid particle with surface $\mathcal{S}_{\mathrm{p}}$ undergoing solid-body motion, i.e. $\mathbf{v}(\mathbf{r})|_{\mathbf{r}\in\mathcal{S}_{\mathrm{p}}} = \mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r}$. Here, $\mathbf{U}$ is a constant translational velocity, and $\boldsymbol{\Omega}$ is a constant angular velocity. Show that under these conditions, we have that $\Phi = \mathbf{F}\cdot\mathbf{U} + \mathbf{T}\cdot\boldsymbol{\Omega}$, with $\mathbf{F}$ and $\mathbf{T}$ the force and torque exerted by the particle on the fluid, respectively. Furthermore, prove that

$$2(\mathbf{F}' \cdot \mathbf{U} + \mathbf{T}' \cdot \boldsymbol{\Omega}) - \Phi' \leq \Phi \leq \Phi^*.$$

(d) Set $\boldsymbol{\Omega} = 0$ and take for the particle a sphere of radius $a$. Let $\mathbf{v}^{(n)}$ be the family of vector fields

$$\mathbf{v}^{(n)} = \left\{ \left[ C\frac{a}{r} + (1-C)\frac{a^n}{r^n} \right] \mathbf{I} + \frac{C}{a^2}\left( \frac{a^3}{r^3} - \frac{a^{n+2}}{r^{n+2}} \right) \mathbf{rr} \right\} \cdot \mathbf{U}, \quad n \geq 2.$$

What should the value of $C$ be such that the corresponding viscous dissipation rate $\Phi^{(n)}$ is an upper bound for $\Phi$? Furthermore, explicitly compute $\Phi^{(2)}$.

(e) Now we establish a lower bound. Consider the stress field

$$\frac{a}{\eta_{\mathrm{s}}}\boldsymbol{\sigma}' = \frac{\alpha}{a^3}\frac{a^5}{r^5}(\mathbf{U}\cdot\mathbf{r})\mathbf{rr} + \beta\left[ \frac{2}{a^3}\frac{a^6}{r^6}(\mathbf{U}\cdot\mathbf{r})\mathbf{rr} - \frac{1}{2a}\frac{a^4}{r^4}(\mathbf{Ur} + \mathbf{rU}) \right].$$

Show that $\boldsymbol{\sigma}'$ satisfies the conditions in (b) for establishing a lower bound on $\Phi$. Explicitly compute this lower bound for arbitrary values of $\alpha$ and $\beta$.

(f) Show that the greatest lower bound occurs for $\alpha = 30/7$ and $\beta = -18/7$. As upper bound we set $\Phi^* = \Phi^{(2)}$. Show that the bounds for the force that the translating sphere exerts on the fluid is

$$\frac{40}{7}\pi\eta_{\mathrm{s}} aU \leq F \leq \frac{56}{9}\pi\eta_{\mathrm{s}} aU,$$

where $|\mathbf{U}| = U$ and $|\mathbf{F}| = F$.

(g) Can we sharpen the upper bound further? If so provide a calculation and discuss your results.

**Problem 3: Properties of the Stokes equations**

Investigate the following phenomena using properties of the Stokes equations.

(a) Consider a solid rod falling under gravity in an unbounded viscous fluid. Show that the rod cannot rotate if no torque is exerted on the particle.

(b) Consider two identical solid spheres settling under gravity in an unbounded viscous fluid. Show that they fall at the same velocity and that their separation remains constant.

(c) Consider a solid particle having a plane of symmetry which is placed into a uniform ambient flow perpendicular to this plane. Show that there is no lift force (i.e., force normal to the flow) exerted on this particle.

**Problem 4: Stokes-flow theorems for general constitutive relations**

In the lecture we have derived three theorems for Stokes flows of Newtonian fluids, related to (i) uniqueness of solutions, (ii) minimum energy dissipation, and (iii) Lorentz reciprocity. Extend these results to general incompressible flows with consitutive relations $\sigma_{\alpha\beta} = -p\delta_{\alpha\beta} + \eta_{\alpha\beta\nu\lambda}\partial_\lambda v_\nu$ with $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$. Here, you cannot assume that the fluid is isotropic. Is microscopic time reversibility essential for these results?