Problem Set 1 – Microhydrodynamics & fluctuations

Problem 1: Warming up with tensors

In this course we write tensors as A with components $A_{\alpha\beta....\gamma\lambda}$, and vectors (rank 1 tensors) as $\mathbf{A}$ with compontents A_α. Generally speaking, greek indices refer to Cartesian components and range over e.g. x, y, z in three-dimensional space. We define the single contraction between two tensors A and B as $[\mathsf{A} \cdot \mathsf{B}]_{\alpha...\beta} = A_{\alpha...\lambda}B_{\lambda...\beta}$ (Einstein convention implied) and the double contraction as $[\mathsf{A} : \mathsf{B}]_{\alpha...\beta} = A_{\alpha...\gamma\lambda}B_{\lambda\gamma...\beta}$. We denote a tensor product without a multiplication sign, i.e. $[\mathsf{AB}]_{\alpha...\beta\gamma...\lambda} = A_{\alpha...\beta}B_{\gamma...\lambda}$.

(a) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, be real vectors. Show that (i) $(\mathbf{ab}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$, (ii) $\mathbf{c} \cdot (\mathbf{ab}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$, and (iii) $(\mathbf{ab}) \cdot (\mathbf{cd}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{ad}$.

(b) Denote the cross product between two tensors as $[\mathsf{A}\times\mathsf{B}]_{\alpha\sigma...\kappa\nu...\mu} = \epsilon_{\alpha\beta\lambda}A_{\beta\sigma...\kappa}B_{\lambda\nu...\mu}$. Here, $\boldsymbol{\epsilon}$ is the Levi-Civita tensor, with the property that $\epsilon_{\alpha\beta\gamma} = [\boldsymbol{\epsilon}]_{\alpha\beta\gamma}$ is fully antisymmetric in all its indices. Can one write $\mathsf{A} \times \mathsf{B}$ as a tensor product of A, B, and $\boldsymbol{\epsilon}$? Do the relations in (a) still hold when one replaces $\cdot$ with $\times$?

Let I be the rank 2 unit tensor, with components $[\mathsf{I}]_{\alpha\beta} = \delta_{\alpha\beta}$.

(c) Show that $\mathsf{I} : (\mathbf{ab}) = \mathrm{tr}(\mathbf{ab})$, where tr(...) denotes the trace.

(d) Express $(\mathbf{a} \times \mathsf{I}) \cdot (\mathbf{b} \times \mathsf{I})$ in terms of tensor products and single contractions only.

(e) Let $\hat{\mathbf{n}}$ be an arbitrary unit vector. Give an interpretation of $\hat{\mathbf{n}}\hat{\mathbf{n}}$ and $\mathsf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}$ viewed as operators acting on vectors with a single contraction.

(f) Denote the orthonormal triad of Cartesian unit vectors by $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ and the set of spherical unit vectors by $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}\}$. Show that $\mathsf{I} = \hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}} = \hat{\mathbf{r}}\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}$.

(g) Compute the following surface integrals over the unit sphere S^2.

$$\int_{S^2} dS\, \mathsf{I}, \quad \int_{S^2} dS\, \hat{\mathbf{r}}\hat{\mathbf{r}}, \quad \int_{S^2} dS\, \hat{\mathbf{r}}\hat{\mathbf{r}}\hat{\mathbf{r}}\hat{\mathbf{r}}, \quad \int_{S^2} dS\, \underbrace{\hat{\mathbf{r}}...\hat{\mathbf{r}}}_{n\,\text{times}}.$$

Problem 2: Dimensionless numbers

Consider the balance of mass and linear momentum for an incompressible Newtonian fluid, i.e. the Navier-Stokes equations. For the moment we assume that there is no body force density acting on the fluid.

(a) By nondimensionalising the Navier-Stokes equations, show that one retrieves the Euler equation for an inviscid fluid when $\mathrm{Re} \gg 1$, with Re the Reynolds number.

(b) What happens when $\mathrm{Re} \ll 1$? Do we necessarily obtain the stationary Stokes equations?

(c) Give a physical interpration for the Reynolds number Re, the Strouhal number Sr, and the oscillatory Reynolds number (also known as the Stokes number) Re_ω.

(d) Suppose now that there is a gravitational force acting on the fluid particles. Show that the importance of gravitational effects is captured by the so-called Froude number $\mathrm{Fr} = U^2/(gL)$, where U are representative speeds, g is the gravitational acceleration, and L is the characteristic length scale in the problem. What kind of physics do you expect as a function of Fr?

Problem 3: Mass conservation law from the Lagrangian description of flows

In the Lecture we have shown how the law of mass conservation in its differential form follows from an Eulerian description of fluids with flow field $\mathbf{v}(\mathbf{r}, t)$. Show that the same equation follows using a Lagrangian description. It can be helpful to consider

$$\frac{d}{dt} \int_{\mathcal{V}(t)} dV \, \rho(\mathbf{r}, t),$$

where $\rho(\mathbf{r}, t)$ is the mass density of the fluid. Furthermore, $\mathcal{V}(t)$ is an arbitrary time-evolving volume where the points on the boundary $\partial\mathcal{V}(t)$ move with velocity $\mathbf{u}(\mathbf{r}, t)$. What happens when $\mathbf{u} = 0$ and $\mathbf{u} = \mathbf{v}$?

Problem 4: Balance of angular momentum

Consider a general (compressible) fluid with mass density $\rho(\mathbf{r}, t)$ and no intrinsic sources of torque. The velocity field is denoted by $\mathbf{v}(\mathbf{r}, t)$.

(a) Show from the balance of angular momentum that the stress tensor must necessarily be symmetric, i.e. $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$.

Now consider a situation where fluid particles are allowed to rotate. The fluid has intrinsic (spin) angular momentum $\mathbf{S}(\mathbf{r}, t) = I\boldsymbol{\omega}(\mathbf{r}, t)$, where I is the average moment of inertia per unit mass and $\boldsymbol{\omega}$ is the mean angular velocity of the fluid particles (not to be confused with the vorticity of the fluid). We denote the orbital angular momentum of the fluid (with respect to the origin) as $\mathbf{L}(\mathbf{r}, t)$.

(b) Derive the balance equation for $\mathbf{S}$ in its differential form. What happens when $\mathbf{S} = 0$? What happens when $\boldsymbol{\sigma}$ is symmetric, but $\mathbf{S} \neq 0$?