Hydrodynamics and Elasticity 2025/2026

Sheet 8

One of the problems will be handed in and marked. If sending solutions over e-mail, please address them to `Agnieszka.Makulska@fuw.edu.pl`

Problem 1 Consider the steady two-dimensional fluid velocity field, given in Cartesian coordinates (x, y) by

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \epsilon & -\gamma \\ \gamma & -\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

where $\epsilon \neq 0$ and $\gamma \neq 0$. Show that the fluid is incompressible. Find the streamfunction ψ and show that the streamlines are defined by

$$(\epsilon - \gamma)(x + y)^2 - (\epsilon + \gamma)(x - y)^2 = C,$$

for C a constant. For each of the three cases below, sketch the streamlines and briefly describe the flow.

(i) $\epsilon = 1,\ \gamma = 0,$

(ii) $\epsilon = 0,\ \gamma = 1,$

(iii) $\epsilon = 1,\ \gamma = 1.$

Problem 2 An ideal fluid is rotating in a gravitational field with a constant angular velocity $\mathbf{\Omega}$, so that the fluid velocity in the lab frame is $\boldsymbol{u} = (-\Omega y, \Omega x, 0)$. Let us find the surfaces of constant pressure, which for a particular choice of $p = p_{\mathrm{atm}}$ will give us the shape of the free surface. According to Bernoulli's law the quantity $p/\rho + \frac{1}{2}u^2 + gz$ is constant, so the surface of constant pressure satisfies the equation

$$z = \mathrm{const} - \frac{\Omega^2}{2g}(x^2 + y^2).$$

But this means that the water level is the highest in the middle of a spinning bucket. What is wrong here? What is the true equation for the constant pressure surface and why?

Problem 3 Consider the purely two-dimensional steady flow of an inviscid incompressible constant density fluid in the absence of body forces. For velocity $\boldsymbol{u}$, the vorticity is $\boldsymbol{\omega} = (0, 0, \omega)$. Let p denote the pressure and ρ the density of the fluid. We define the streamfunction $\mathbf{\Psi}(x, y) = \psi(x, y)\boldsymbol{e}_z$ such that $\boldsymbol{u}(x, y) = \nabla \times \mathbf{\Psi}$. Show that if ω is a constant both in space and time, then

$$\frac{|\boldsymbol{u}|^2}{2} + \omega\psi + \frac{p}{\rho} = C,$$

where C is a constant.

Now consider a fluid with constant (in both space and time) vorticity ω in the cylindrical annular region $a < r < 2a$. The streamlines are concentric circles, with the fluid speed zero on $r = 2a$ and $V > 0$ on $r = a$. Calculate the velocity field, and hence show that

$$\omega = -\frac{2V}{3a}.$$

Deduce that the pressure difference between the outer and inner edges of the annular region is

$$\Delta p = \left(\frac{15 - 16 \ln 2}{18}\right) \rho V^2$$

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