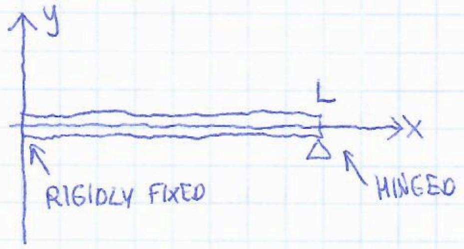
DAWID BUGAJEWSKI

PROBLEM 3

Well done! 10/10



DEFORMED SHAPE = ?

MAXIMAL DEFLECTION – WHERE & VALUE?

THE SHAPE $y(x)$ OF THE ROD IS DESCRIBED BY THE DIFFERENTIAL EQUATION:

$$EI\frac{d^4y}{dx^4} = K \quad \text{, WHERE}$$

- E – YOUNG'S MODULUS
- I – AREA MOMENT
- K – BODY FORCE PER UNIT LENGTH ALONG y (HERE IT WILL BE $\frac{-Mg}{L}$, M – MASS OF THE ROD)

THEREFORE

$$\frac{d^4y}{dx^4} = \frac{K}{EI} = \text{CONST.}$$, SO BY INTEGRATING FOUR TIMES,

$$y(x) = A + Bx + Cx^2 + Dx^3 + \frac{K}{24EI}x^4 \qquad \left[(x^4)'''' = (4x^3)''' = (12x^2)'' = (24x)' = 24\right]$$

(A, B, C, D – SOME CONSTANTS)

TO DETERMINE A, B, C, D, WE HAVE TO IMPOSE SOME BOUNDARY CONDITIONS:

- $y(0) = 0 \Rightarrow A = 0$
- $y'(0) = 0 \Rightarrow B = 0$

(SINCE THE ROD IS RIGIDLY FIXED AT THE LEFT END)

- $y(L) = 0$
- $y''(L) = 0$

(SINCE THE ROD IS SIMPLY SUPPORTED AT THE RIGHT END, $y(L) = 0$ AND, SINCE THE HINGE ALLOWS THE ROD TO ROTATE, IN EQUILIBRIUM THE TORQUE, WHICH IS AT POINT x PROPORTIONAL TO $y''(x)$, MUST VANISH)

THEREFORE $y(x) = Cx^2 + Dx^3 + \frac{K}{24EI}x^4$, WHERE

$$\begin{cases} 0 = y(L) = CL^2 + DL^3 + \frac{K}{24EI}L^4 \\ 0 = y''(L) = 2C + 6DL + \frac{K}{2EI}L^2 \end{cases}$$

$(x^2)'' = 2$
$(x^3)'' = 6x$
$(x^4)'' = 12x^2$

LET'S SOLVE THIS LINEAR SYSTEM:

$$\begin{cases} 0 = C + DL + \frac{K}{24EI}L^2 \\ 0 = C + 3DL + \frac{K}{4EI}L^2 \end{cases}$$

$$0 = -2DL - \frac{K}{EI}L^2\left(\frac{1}{4} - \frac{1}{24}\right)$$

$$2DL = -\frac{K}{EI}\frac{5}{24}L^2$$

$$D = \frac{-5KL}{48EI}$$

$$C = -DL - \frac{K}{24EI}L^2 = \frac{5KL^2}{48EI} - \frac{2KL^2}{48EI} = \frac{3KL^2}{48EI}$$

THERE FORE

$$y(x) = \frac{3KL^2}{48EI}x^2 - \frac{5KL}{48EI}x^3 + \frac{K}{24EI}x^4 =$$
$$= \frac{K}{48EI}x^2\left(3L^2 - 5Lx + 2x^2\right)$$

(WHERE $K = -\frac{Mg}{L} < 0$, E - YOUNG'S MODULUS, $I = \int_{\text{CROSS SECTION}} x^2 dS = \frac{1}{2}\int_{-\parallel-} r^2 dS = \frac{1}{2}\cdot 2\pi \int_0^R r r^2 dS = \frac{\pi R^4}{4}$ (R - RADIUS OF THE ROD),

OF COURSE IF THE CROSS SECTION OF THE ROD IS NOT CIRCULAR, THE RESULT FOR $y(x)$ IS STILL VALID, JUST THE FORMULA FOR I WILL BE DIFFERENT)

LET'S FIND THE POSITION AND THE VALUE OF MAXIMAL DEFLECTION:

$$y'(x) = \frac{K}{48EI}\left(6L^2x - 15Lx^2 + 8x^3\right)$$

WHEN IS IT EQUAL TO 0?

$x(6L^2 - 15Lx + 8x^2) = 0$

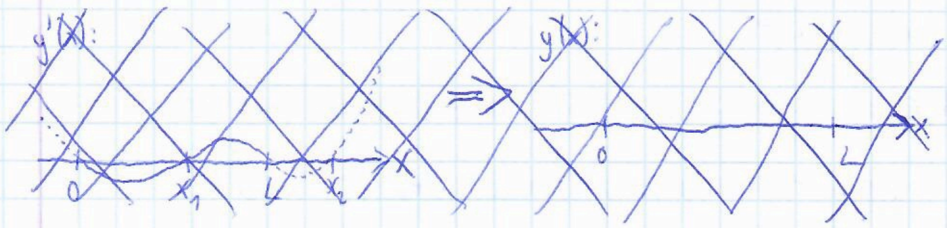
$x = 0$ OR $\quad 8x^2 - 15Lx + 6L^2 = 0$

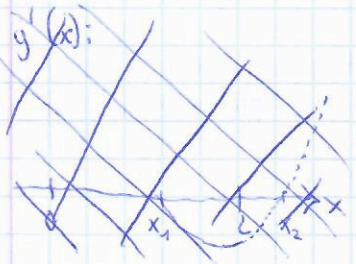
END OF THE ROD - NOT WHAT WE'RE LOOKING FOR

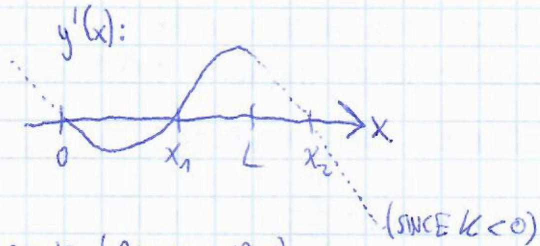
$\Delta = (-15L)^2 - 4\cdot 8\cdot 6L^2 = 225L^2 - 192L^2 = 123L^2 - 90L^2 = 33L^2$

$x_1 = \frac{15L - L\sqrt{33}}{16} = L\frac{15-\sqrt{33}}{16} \in [0, L)$ - OK.

$x_2 = \frac{15L + L\sqrt{33}}{16} > \frac{15L + L}{16} = L$ - NOT PHYSICAL ($x \in [0, L]$)

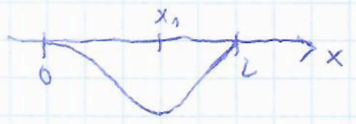






THEREFORE THE SHAPE OF THE ROD IS ($\approx$ OF COURSE)

y(x):



THEREFORE THE POSITION OF MAXIMAL DEFLECTION IS $x_0 = L\frac{15-\sqrt{33}}{16}$ AND ITS VALUE IS

$$y(x_0) = \frac{KL^2}{48EI}\left(\frac{15-\sqrt{33}}{16}\right)^2\left(3L^2 - 5L^2\frac{15-\sqrt{33}}{16} + 2\left(\frac{15-\sqrt{33}}{16}\right)^2\right)$$

$$= \frac{KL^4}{48EI}\left(\frac{225 - 30\sqrt{33} + 33}{256}\right)\left(3 - \frac{75 - 5\sqrt{33}}{16} + \frac{225 + 33 - 30\sqrt{33}}{128}\right)$$

$$= \frac{KL^4}{48EI}\,\frac{258 - 30\sqrt{33}}{256}\,\frac{384 - 600 + 40\sqrt{33} + 258 - 30\sqrt{33}}{128} = \frac{KL^4}{48EI}\,\frac{258 - 30\sqrt{33}}{256}\,\frac{42 + 10\sqrt{33}}{128} =$$

$$= \frac{KL^4}{EI}\,\frac{936 + 1320\sqrt{33}}{3\cdot 2^{19}}$$

33
11
258
42
516
1032
10836
300·33 = 9900
30·42 = 1260