Hydrodynamics and Elasticity
Test I, Problem 2

A cylider of radius R and infinite length is rotating about its axis (z) with a constant angular velocity ω. The material of the cylinder has the density ρ_0 and is characterised by Lamé constants λ and μ.

(a) Derive the formula for the divergence of the deformation field $u_r(r)$, where r is the radius in cylindrical coordinates.

(b) Find the deformation field assuming that the outer wall of the cylinder is stress-free. Find the distribution of stresses arising from the rotation.

(c) Interpret the obtained components of the strain tensor. Is the material of the cylidner stretched or compressed in the radial direction (i.e. along $\boldsymbol{e}_r$)? Is there a point at which the material is neither stretched, nor compressed?

(d) If the cylinder is rotating fast enough, the resulting stresses may lead to the material breaking. If this is the case, where can we expect a crack to appear?

Solution

(a) The force density is given by $\boldsymbol{f} = \rho_0\omega^2 r\boldsymbol{e}_r$. From symmetry, we infer that the cylinderr undergoes purely radial deformation

$$\boldsymbol{u} = u_r(r)\boldsymbol{e}_r \tag{1}$$

As discussed in classes, the divergence reads then $\nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r)$.

(b) In classes we found the Navier equation for this particular form of deformation, and in the presence of a radial body force density f_r it takes the form

$$(\lambda+2\mu)\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru_r)\right] = -\rho_0 f_r \tag{2}$$

In the non-inertial frame of the cylinder, we have the centrifugal force $f_r = \omega^2 r$. Thus the Navier equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru_r)\right] = -\frac{\rho_0\omega^2}{\lambda+2\mu}r \tag{3}$$

We find the general form of deformation by integration

$$u_r = Ar + \frac{B}{r} - \frac{\rho_0\omega^2}{8(\lambda+2\mu)}r^3. \tag{4}$$

Using the technique discussed in classes, we find the form of the strain tensor $\widehat{\boldsymbol{E}}$. Its only non-vanishing components are

$$E_{rr} = \frac{\mathrm{d}u_r}{\mathrm{d}r} = A - \frac{B}{r^2} - \frac{3\rho_0\omega^2}{8(\lambda+2\mu)}r^2, \qquad E_{\varphi\varphi} = \frac{u_r}{r} = A + \frac{B}{r^2} - \frac{\rho_0\omega^2}{8(\lambda+2\mu)}r^2. \tag{5}$$

Using Hooke's law

$$\widehat{\boldsymbol{T}} = 2\mu\widehat{\boldsymbol{E}} + \lambda(\mathrm{Tr}\widehat{\boldsymbol{E}})\mathbf{1}, \tag{6}$$

we find the non-vanishing components of the stress tensor

$$T_{rr} = (\lambda+2\mu)\frac{\partial u_r}{\partial r} + \lambda\frac{u_r}{r}, \qquad T_{\varphi\varphi} = (\lambda+2\mu)\frac{u_r}{r} + \lambda\frac{\mathrm{d}u_r}{\mathrm{d}r}, \qquad T_{zz} = \lambda\left(\frac{u_r}{r} + \frac{\mathrm{d}u_r}{\mathrm{d}r}\right). \tag{7}$$

Since the deformation has to be finite at $r = 0$, we find $B = 0$. We need to determine A, for which we require the stress-free condition at the outer surface of the cylinder:

$$T_{rr}|_{r=R} = 0. \tag{8}$$

This determines the constant A, so the final solution reads:

$$u_r(r) = \frac{1}{8}\frac{\rho_0\omega^2R^2}{\lambda+2\mu}r\left(\frac{3\mu+2\lambda}{\mu+\lambda} - \frac{r^2}{R^2}\right). \tag{9}$$

From that we find the components of $\widehat{\boldsymbol{E}}$:

$$E_{rr} = \frac{1}{8}\frac{\rho_0\omega^2R^2}{\lambda+2\mu}\left[\frac{3\mu+2\lambda}{\mu+\lambda} - 3\frac{r^2}{R^2}\right], \qquad E_{\varphi\varphi} = \frac{1}{8}\frac{\rho_0\omega^2R^2}{\lambda+2\mu}\left(\frac{3\mu+2\lambda}{\mu+\lambda} - \frac{r^2}{R^2}\right). \tag{10}$$

(c) From the analysis of E_{rr} we see that for $r < x = R\sqrt{\frac{1}{3}\frac{3\mu+2\lambda}{\mu+\lambda}}$ the material is stretched ($E_{rr} > 0$), for $r > x$ the material is compressed, and at $r = x$ it is neither stretched, nor compressed in this direction. But $E_{\varphi\varphi} > 0$, so the material is stretched in the azimuthal direction.

(d) At $r = 0$ the stresses are maximal, so this is where a crack would first appear.