MATEUSZ PAŃTAK ~~10~~ 10 pkt

Problem 3

We start with Navier-Stokes equation (6) for incompressible fluids:

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}\right) = -\nabla p + \mu\nabla^2\bar{u} + \bar{f}$$

By symmetry we can say: u_x depends only on z and t so $(\bar{u}\cdot\nabla)\bar{u} = u_x\frac{\partial u_x}{\partial x} = 0$

Also we neglect $\bar{f} = 0$. And now let's look on our equation on component x:

$$\rho\frac{\partial u_x}{\partial t} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u_x}{\partial z^2}$$ ✓

gradient

Substituting the complex form of the pressure from hint and assuming a similar response for velocity: $u_x(z,t) = u(z)e^{-i\omega t}$, then: ✓

$$\rho(-i\omega)u(z)\cancel{e^{-i\omega t}} = -p_0\cancel{e^{-i\omega t}} + \mu\frac{d^2u}{dz^2}\cancel{e^{-i\omega t}} \quad \Big| : e^{-i\omega t}$$

By this we obtain standard ODE: $\frac{d^2u}{dz^2} + \frac{i\omega\rho}{\mu}u = \frac{p_0}{\mu}$ (nie wiem po angielsku ?)

In this case the general solution consists of homogenous part and szczególne rozwiązanie. (particular solution)

1. Szczególne rozwiązanie: for constant term $\frac{p_0}{\mu}$ we have:

$$u_s = \frac{p_0}{\mu k^2} = \frac{p_0}{\mu\left(i\omega\frac{\rho}{\mu}\right)} = \frac{p_0}{i\omega\rho}$$

2. Homogenous solution is obvious: $u_h(z) = C_1\cos(kz) + C_2\sin(kz)$

$\Longrightarrow$ The general solution $u(z) = C_1\cos(kz) + C_2\sin(kz) + \frac{p_0}{i\omega\rho}$

$u(z) = u_s + u_h(z)$ now we need to apply boundary conditions

Boundary condition:

Symmetry tells that flow must be symmetric with respect to the center plane $(z=0)$.
Therefore the odd function $\sin(kz)$ must vanish $\Longrightarrow C_2 = 0$.

∘ No-slip condition: velocity at walls is zero:

$u\left(\pm\frac{h}{2}\right) = 0$ substituting $z = \frac{h}{2}$;

$$C_1 \cos\left(k\frac{h}{2}\right) + \frac{p_0}{i\omega\rho} = 0 \Longrightarrow C_1 = -\frac{p_0}{i\omega\rho\cos\left(k\frac{h}{2}\right)}$$ ✓

Combining this we have: $u(z) = \frac{p_0}{i\omega\rho}\left(1 - \frac{\cos(kz)}{\cos\left(k\frac{h}{2}\right)}\right)$. ok

Now mean flow over the cross-section:

$$\bar{u}(t) = \left(\frac{1}{h}\int_{-h/2}^{h/2} u(z)\,dz\right) e^{-i\omega t}$$

(*) Let's notice $u(z)$ is even function so we can integrate from 0 to $\frac{h}{2}$ and multiply result by factor 2:

$$\bar{u} = \frac{2}{h}\int_0^{h/2} \frac{p_0}{i\omega\rho}\left(1 - \frac{\cos(kz)}{\cos\left(k\frac{h}{2}\right)}\right)dz = \left\{\int \cos(kz)\,dz = \frac{1}{k}\sin(kz)\right\} =$$

$$= \frac{2}{h}\frac{p_0}{i\omega\rho}\left[z - \frac{\sin(kz)}{k\cos\left(k\frac{h}{2}\right)}\right]_0^{h/2} = \frac{2}{h}\frac{p_0}{i\omega\rho}\left(\frac{h}{2} - \frac{\sin(kh/2)}{k\cos(kh/2)}\right)$$

$$\Longrightarrow \bar{u} = \frac{p_0}{i\omega\rho}\left(1 - \frac{\operatorname{tg}\left(\frac{kh}{2}\right)}{\frac{kh}{2}}\right)$$ ✓

We know that $k = \sqrt{\frac{i\omega}{\nu}}$, let's define $\alpha = \frac{kh}{2} = \frac{h}{2}\sqrt{\frac{i\omega}{\nu}}$

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Problem 3 (continuation)

Then: $\bar{u}(t) = \frac{p_0}{i\omega\rho}\left(1 + \frac{tg(\alpha)}{\alpha}\right)e^{-i\omega t}$

Case 1: Low frequency: $\omega << \frac{\nu}{h^2} \implies |\alpha| << 1$

It is slow oscillation where viscous force will dominate. Let's expand $tg(\alpha)$ in Taylor:

$tg(\alpha) \approx \alpha + \frac{\alpha^3}{3} + \dots$ so $1 - \frac{tg(\alpha)}{\alpha} \approx 1 - \frac{\alpha + \frac{\alpha^3}{3}}{\alpha} = 1 - \left(1 + \frac{\alpha^2}{3}\right) = -\frac{\alpha^2}{3}$

Then $\alpha^2 = \frac{k^2 h^2}{4} = \frac{i\omega h^2}{4\nu}$ so $\bar{u} = \frac{p_0}{i\omega\rho}\left(-\frac{i\omega h^2}{12\nu}\right) = -\frac{p_0 h^2}{12\mu}$

$\implies \bar{u}(t) = -\frac{h^2}{12\mu} p_0 e^{-i\omega t}$ ✓

The flow is in phase with pressure (the sign shows it is opposite to the pressure gradient: flows from high to low pressure).

Case 2: High frequency: $\omega >> \frac{\nu}{h^2} \implies |\alpha| >> 1$

It is fast oscillation where inertia will dominate. For large arguments (imaginary)

$tg(\alpha) \to i$ so $\frac{tg(\alpha)}{\alpha} \to 0$, then $\bar{u}(t) \approx \frac{p_0}{i\omega\rho}$ ✓

The equation becomes $\rho\frac{\partial\bar{u}}{\partial t} = -\frac{\partial p}{\partial x} \to$ behavior of an ideal fluid - inviscid

The viscosity term becomes negligible compared to inertia. The velocity lags the pressure gradient by factor $\frac{1}{i}$ so 90°.

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