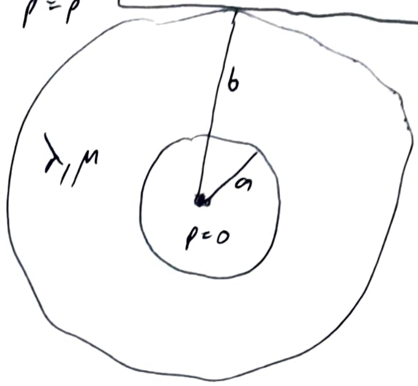
1.

David Tuthan

10



We expect the deformation to only have the r component, dependent only on „r":

$\vec{u} = u_r(r)\,\hat{e_r}$

We use Navier-Cauchy equation, to find $u_r(r)$:

$\vec{f} + \mu\bar{\nabla}^2\bar{u} + (\lambda+\mu)\bar{\nabla}(\bar{\nabla}\cdot\bar{u}) = 0$

$\vec{f} = 0$

$[\bar{\nabla}\times(\bar{\nabla}\times\bar{u})]_i = \varepsilon_{ijk}\nabla_j(\bar{\nabla}\times\bar{u})_k = \varepsilon_{ijk}\nabla_j\varepsilon_{klm}\nabla_l u_m = \varepsilon_{kij}\varepsilon_{klm}\nabla_j\nabla_l u_m =$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\nabla_j\nabla_l u_m = \nabla_j\nabla_i u_j - \nabla_j\nabla_j u_i = \nabla_i(\bar{\nabla}\cdot\bar{u}) - \bar{\nabla}^2 u_i$

$\Rightarrow \bar{\nabla}^2\bar{u} = \bar{\nabla}(\bar{\nabla}\cdot\bar{u}) - \bar{\nabla}\times(\bar{\nabla}\times\bar{u})$

In our case the deformation is clearly irrotational $\nabla\times\bar{u} = 0$

$\Rightarrow \bar{\nabla}^2\bar{u} = \bar{\nabla}(\bar{\nabla}\cdot\bar{u})$ ✓

And N-C equation simplifies:

$(\lambda+2\mu)\bar{\nabla}(\bar{\nabla}\cdot\bar{u}) = 0$

$\Rightarrow \bar{\nabla}(\bar{\nabla}\cdot\bar{u}) = 0 \Rightarrow \bar{\nabla}\cdot\bar{u} = C_1$

We know, that $Tr(\bar{\nabla}\otimes\bar{u}) = \bar{\nabla}\cdot\bar{u}$, so we use the hint:

$\bar{\nabla}\cdot\bar{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}$

We have only $u_r(r)$:

$\bar{\nabla}\cdot\bar{u} = \frac{\partial u_r}{\partial r} + \frac{2u_r}{r} = C_1 \;\Big|\cdot r^2$

$\frac{\partial}{\partial r}\left(\frac{u_r}{r}\right) = \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{1}{r^2}u_r$

$\frac{\partial}{\partial r}(r^2u_r) = 2ru_r + r^2\frac{\partial u_r}{\partial r}$

$\frac{\partial}{\partial r}\left(\frac{u_r}{r^2}\right) = \frac{1}{r^2}\frac{\partial u_r}{\partial r} - \frac{2}{r^3}u_r$

$r\frac{\partial u_r}{\partial r} + 2u_r$

$$\frac{\partial u_r}{\partial r} + 2 r u_r = C_1 r^2$$

$$\frac{\partial}{\partial r}\left(r^2 u_r\right) = C_1 r^2 \quad \Big/ \int dr$$

$$r^2 u_r = \frac{C_1}{3} r^3 + C_2$$

$$u_r = \frac{C_1}{3} r + \frac{C_2}{r^2} \quad \checkmark$$

We redefine the constant C_1:

$$\boxed{u_r = C_1 r + \frac{C_2}{r^2}}$$

We have to calculate strain tensor $\tilde{\varepsilon} = (\tilde{\nabla} \bar{u})^s$:

$$\tilde{\nabla}\bar{u} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & 0 & 0 \\ 0 & \frac{u_r}{r} & 0 \\ 0 & 0 & \frac{u_r}{r} \end{bmatrix} = \tilde{\varepsilon}$$

already symmetric!

$$\tilde{\varepsilon} = \begin{bmatrix} C_1 - \frac{2C_2}{r^3} & 0 & 0 \\ 0 & C_1 + \frac{C_2}{r^3} & 0 \\ 0 & 0 & C_1 + \frac{C_2}{r^3} \end{bmatrix}$$

Now stress tensor:

$$\tilde{T} = 2\mu\tilde{\varepsilon} + \lambda \, Tr(\tilde{\varepsilon}) \mathbb{1} \qquad Tr(\tilde{\varepsilon}) = 3C_1$$

$$\tilde{T} = 2\mu \begin{bmatrix} C_1 - \frac{2C_2}{r^3} & 0 & 0 \\ 0 & C_1 + \frac{C_2}{r^3} & 0 \\ 0 & 0 & C_1 + \frac{C_2}{r^3} \end{bmatrix} + 3\lambda C_1 \mathbb{1}$$

Now we apply the boundary conditions:

$$\left.\begin{aligned} \tilde{T}\hat{e_r}\Big|_{r=b} &= -p\,\hat{e_r} \\ \tilde{T}(-\hat{e_r})\Big|_{r=a} &= 0 \cdot \hat{e_r} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} T_{rr}\Big|_{r=b} &= -p \\ T_{rr}\Big|_{r=a} &= 0 \end{aligned}\right\}$$

$$2\mu\left(C_1 - \frac{2C_2}{a^3}\right) + 3\lambda C_1 = 0 \quad \text{and} \quad 2\mu\left(C_1 - \frac{2C_2}{b^3}\right) + 3\lambda C_1 = -p$$

$$\frac{a^3-b^3}{a^3b^3}$$

$$-\frac{4\mu C_2}{a^3}+\frac{4\mu C_2}{b^3}=p \Rightarrow C_2\left[4\mu\left(\frac{1}{b^3}-\frac{1}{a^3}\right)\right]=p \Rightarrow C_2=\frac{p}{4\mu}\frac{1}{\left(\frac{1}{b^3}-\frac{1}{a^3}\right)}$$

$$\Rightarrow \boxed{C_2=\frac{pa^3b^3}{4\mu(a^3-b^3)}}$$

V

$$C_1(2\mu+3\lambda)=\frac{4\mu C_2}{a^3} \Rightarrow C_1=\frac{4\mu C_2}{(2\mu+3\lambda)a^3}$$

$$\Rightarrow \boxed{C_1=\frac{pb^3}{(2\mu+3\lambda)(a^3-b^3)}}$$

V

$$u_r=\frac{pb^3}{(2\mu+3\lambda)(a^3-b^3)}r+\frac{pa^3b^3}{4\mu(a^3-b^3)}\frac{1}{r^2}$$

$$\lambda=\frac{E\nu}{(1-2\nu)(1+\nu)} \qquad \mu=\frac{E}{2(1+\nu)} \qquad 2\mu+3\lambda=\frac{2E}{2(1+\nu)}+\frac{3E\nu}{(1-2\nu)(1+\nu)}=$$

$$=\frac{2(1-2\nu)+3\nu\cdot 2}{2(1-2\nu)(1+\nu)}E=E\frac{2-4\nu+6\nu}{2(1-2\nu)(1+\nu)}=E\frac{2\nu+2}{2(1-2\nu)(1+\nu)}=\frac{E}{(1-2\nu)}$$

$$u_r=\frac{pb^3(1-2\nu)}{E(a^3-b^3)}r+\frac{pa^3b^3(1+\nu)}{2E(a^3-b^3)}\frac{1}{r^2}$$ V

B)

$$u_r(r=b)=\frac{pb^4(1-2\nu)}{E(a^3-b^3)}+\frac{pa^3b(1+\nu)}{2E(a^3-b^3)}=\frac{pb\left[2b^3(1-2\nu)+a^3(1+\nu)\right]}{2E(a^3-b^3)}$$

$$u_r(r=a)=\frac{pab^3(1-2\nu)}{E(a^3-b^3)}+\frac{pab^3(1+\nu)}{2E(a^3-b^3)}=\frac{pab^3\left[2(1-2\nu)+1+\nu\right]}{2E(a^3-b^3)}$$

$$\Delta D=u_r(r=b)-u_r(r=a)=\frac{pb}{2E(a^3-b^3)}\left[2b^3(1-2\nu)+a^3(1+\nu)-2ab^2(1-2\nu)-ab^2(1+\nu)\right]=$$

$b=2a$

$$=\frac{2pa}{2E(a^3-8a^3)}\left[16a^3(1-2\nu)+a^3(1+\nu)-2a\cdot 4a^2(1-2\nu)-4a^3(1+\nu)\right]=$$

$$=\frac{p}{-7Ea^2}\left[16a^3(1-2\nu)-8a^3(1-2\nu)+a^3(1+\nu)-4a^3(1+\nu)\right]=$$

$$-16$$
$$-3$$

$$=\frac{p}{-7E}\left[8a(1-2\nu)-3a(1+\nu)\right]=\frac{-p}{7E}\left[5a-19a\nu\right]=\frac{pa}{7E}(19\nu-5)$$ V

3/4

will increase for $\nu > \frac{5}{19}$.

...ally $\nu \in [0, \frac{1}{2}]$ so it is possible.

c)

$$\tau_{rr} = 2\mu\left(C_1 - \frac{2C_2}{r^3}\right) + 3\lambda C_1 = (2\mu + 3\lambda) C_1 - \frac{4\mu C_2}{r^3} =$$

$$= \frac{p b^3}{(a^3 - b^3)} - \frac{p a^3 b^3}{(a^3 - b^3)} \frac{1}{r^3} = \frac{p b^3}{(a^3 - b^3)} \left[1 - \frac{a^3}{r^3}\right]$$

The τ_{rr} component doesnt depend on „r".