Tensors and their transformations

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I. COMPONENTS OF A TENSOR

Consider a vector $\boldsymbol{a}$ and a tensor $\boldsymbol{T}$, which transforms the vector $\boldsymbol{a}$ into $\boldsymbol{b}$, that is

$$\boldsymbol{b} = \boldsymbol{T}\boldsymbol{a}. \tag{1}$$

In the Cartesian basis of $\mathbb{R}^3$, $\boldsymbol{a}$ has a decomposition

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3, \tag{2}$$

similarly to $\boldsymbol{b}$. We would like to find the components of $\boldsymbol{T}$. We have

$$\boldsymbol{b} = \boldsymbol{T}\boldsymbol{a} = a_1\boldsymbol{T}\boldsymbol{e}_1 + a_2\boldsymbol{T}\boldsymbol{e}_2 + a_3\boldsymbol{T}\boldsymbol{e}_3. \tag{3}$$

From the above, multiplying by the respective unit vectors, we get the components of $\boldsymbol{b}$

$$b_1 = \boldsymbol{e}_1\cdot\boldsymbol{b} = a_1\boldsymbol{e}_1\cdot\boldsymbol{T}\boldsymbol{e}_1 + a_2\boldsymbol{e}_1\cdot\boldsymbol{T}\boldsymbol{e}_2 + a_3\boldsymbol{e}_1\cdot\boldsymbol{T}\boldsymbol{e}_3, \tag{4}$$
$$b_2 = \boldsymbol{e}_2\cdot\boldsymbol{b} = a_1\boldsymbol{e}_2\cdot\boldsymbol{T}\boldsymbol{e}_1 + a_2\boldsymbol{e}_2\cdot\boldsymbol{T}\boldsymbol{e}_2 + a_3\boldsymbol{e}_2\cdot\boldsymbol{T}\boldsymbol{e}_3, \tag{5}$$
$$b_3 = \boldsymbol{e}_3\cdot\boldsymbol{b} = a_1\boldsymbol{e}_3\cdot\boldsymbol{T}\boldsymbol{e}_1 + a_2\boldsymbol{e}_3\cdot\boldsymbol{T}\boldsymbol{e}_2 + a_3\boldsymbol{e}_3\cdot\boldsymbol{T}\boldsymbol{e}_3. \tag{6}$$

We call the subsequent products the matrix elements T_{ij} of the tensor $\boldsymbol{T}$ and write the above equation as

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \tag{7}$$

which in the matrix form is simply

$$[\boldsymbol{b}] = [\boldsymbol{T}][\boldsymbol{a}]. \tag{8}$$

Using index notation, we can write the above as

$$b_i = T_{ij}a_j. \tag{9}$$

Note: we assume a **convention**, according to which the tensor $\boldsymbol{T}$ acts on the unit vectors of the Cartesian basis as

$$\boldsymbol{T}\boldsymbol{e}_i = T_{ji}\boldsymbol{e}_j, \tag{10}$$

so that $\boldsymbol{T}\boldsymbol{e}_1 = T_{11}\boldsymbol{e}_1 + T_{21}\boldsymbol{e}_2 + T_{31}\boldsymbol{e}_3$, etc. We take up this convention, because then we can write the m-th component of $\boldsymbol{b}$ as

$$b_m = \boldsymbol{b}\cdot\boldsymbol{e}_m = a_iT_{ji}\boldsymbol{e}_j\cdot\boldsymbol{e}_m = a_iT_{ji}\delta_{jm} = T_{mi}a_i, \tag{11}$$

which corresponds to the matrix relationship (8). Had we decided otherwise, that is

$$\boldsymbol{T}\boldsymbol{e}_i = T_{ij}\boldsymbol{e}_j, \tag{12}$$

then the matrix equation we would get (it is easy to check) would have the form

$$[\boldsymbol{b}] = [\boldsymbol{T}]^\top[\boldsymbol{a}]. \tag{13}$$

This would correspond to a tensor equation $\boldsymbol{b} = \boldsymbol{T}\boldsymbol{a}$, which is less natural.

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II. ORTHOGONAL TRANSFORMS

We define an orthogonal transform to be one that does not change the length or angles between vectors. The tensor $\boldsymbol{Q}$ which corresponds to such a transformation, has the following property

$$\boldsymbol{Q}\boldsymbol{Q}^{\mathsf{T}} = \boldsymbol{Q}^{\mathsf{T}}\boldsymbol{Q} = \mathbf{1}. \tag{14}$$

It follows that the determinant of a matrix that corresponds to an orthogonal transformation must be equal to

$$\det\boldsymbol{Q} = \pm 1, \tag{15}$$

Note that any two Cartesian frames of reference $\{\boldsymbol{e}'\}$ and $\{\boldsymbol{e}\}$ are related by an orthogonal transformation $\boldsymbol{Q}$ such that

$$\boldsymbol{e}'_i = \boldsymbol{Q}\boldsymbol{e}_i, \tag{16}$$

so, according to our convention

$$\boldsymbol{e}'_i = Q_{ji}\boldsymbol{e}_j. \tag{17}$$

It is now easy to check that the components of $[\boldsymbol{Q}]$ are given by

$$Q_{mn} = \cos(\boldsymbol{e}_m,\ \boldsymbol{e}'_n), \tag{18}$$

so they are cosines of the angles between unit vectors of the 'old' and 'new' basis. This matrix of cosines is called the transformation matrix from the non-primed to the primed frane. One can check that if we can express new basis unit vectors $\boldsymbol{e}'_i$ in terms of old basis vectors $\boldsymbol{e}_i$, then we can readily write the matrix $\boldsymbol{Q}$, columns of which are new vectors (expressed in terms of old vectors)

$$\boldsymbol{Q} = \left(\begin{array}{c|c|c} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 & \boldsymbol{e}'_3 \end{array} \right). \tag{19}$$

III. DEFINITION OF TENSORS THROUGH THEIR TRANSFORMATION RULES

a. Transformation of Cartesian components of a vector Consider a transformation of the Cartesian components of a vector $\boldsymbol{a}$ between two bases $\{\boldsymbol{e}\}$ and $\{\boldsymbol{e}'\}$. In both, the vector has a decomposition

$$\boldsymbol{a} = a_i\boldsymbol{e}_i = a_j\boldsymbol{e}'_j. \tag{20}$$

According to our convention, we know how basis vectors transform:

$$\boldsymbol{e}'_i = Q_{mi}\boldsymbol{e}_m, \tag{21}$$

so that

$$a'_i = \boldsymbol{a}\cdot Q_{mi}\boldsymbol{e}_m = Q_{mi}a_m. \tag{22}$$

In the matrix notation, this means

$$[\boldsymbol{a}]' = [\boldsymbol{Q}]^{\mathsf{T}}[\boldsymbol{a}]. \tag{23}$$

Note: We need to distinguish two objects here: the equation above concerns the *same* vector, represented in a different basis. It is *not* equivalent to $\boldsymbol{a}' = \boldsymbol{Q}^{\mathsf{T}}\boldsymbol{a}$, in which $\boldsymbol{a}$ and $\boldsymbol{a}'$ are different vectors related by the action of a tensor $\boldsymbol{Q}^{\mathsf{T}}$.

b. Transformation of Cartesian components of a tensor We can similarly show, that Cartesian components of a tensor $\boldsymbol{T}$ in two bases can be related by the following transformation

$$T'_{ij} = \boldsymbol{e}'_i\cdot\boldsymbol{T}\boldsymbol{e}'_j = Q_{mi}\boldsymbol{e}_m\cdot\boldsymbol{T}Q_{nj}\boldsymbol{e}_n = Q_{mi}Q_{nj}\boldsymbol{e}_m\boldsymbol{T}\boldsymbol{e}_n = Q_{mi}Q_{nj}T_{mn}, \tag{24}$$

which can be written in a matrix form as

$$[\boldsymbol{T}]' = [\boldsymbol{Q}]^{\mathsf{T}}[T][\boldsymbol{Q}]. \tag{25}$$

Of course, the previous **Note** holds sway, so this equation is *not* equivalent to writing $\boldsymbol{T}' = \boldsymbol{Q}^{\mathsf{T}}\boldsymbol{T}\boldsymbol{Q}$, which relates two tensors $\boldsymbol{T}$ i $\boldsymbol{T}'$, and not components of the same tensor in different bases.

c. Transformation properties and tensorial character From the considerations above we conclude that to characterise a vector or tensor quantity, it is enough to know its components in a certain basis, because then we know how to transform them to a different basis. We can generally classify different objects according to how they transform under a change of basis. Consider again two bases $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$ and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, along with an orthogonal transformation $\boldsymbol{e}'_i = \boldsymbol{Q}\boldsymbol{e}_i$.

We define the Cartesian components of tensors:

$\alpha' = \alpha$	scalar (tensor of rank 0)
$a'_i = Q_{mi}a_m$	vector (tensor of rank 1)
$T'_{ij} = Q_{mi}Q_{nj}T_{mm}$	tensor (tensor of rank 2)
$D'_{ijk} = Q_{mi}Q_{nj}Q_{pk}D_{mnp}$	tensor of rank 3
$C'_{ijkl} = Q_{mi}Q_{nj}Q_{pk}Q_{ql}C_{mnpq}$	tensor of rank 4