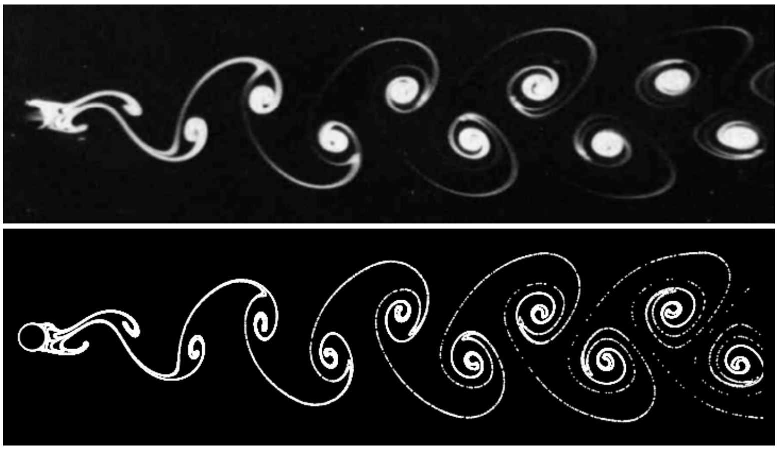
Hydrodynamics and Elasticity 2023/2024

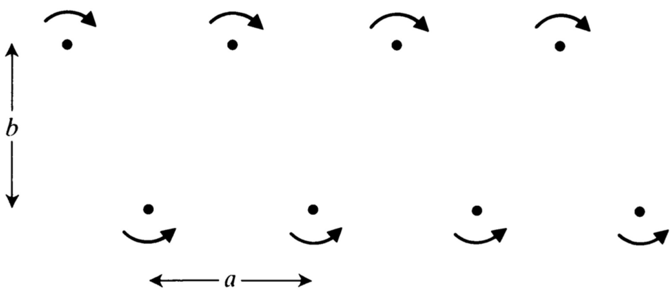
Christmas Problem Sheet

Here are a few problems for you for Christmas to keep your mind occupied during lengthy family dinners. The problems are not obligatory, but you can make yourself a Christmas present by getting five extra points for the solutions.

❄ **Problem 1** A von Karman vortex street is the repeating pattern in parallel rows of vortices that form in the wake of an obstruction in flowing fluid.

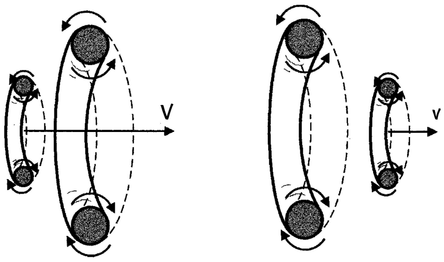


A mathematical model of such a vortex street, introduced by von Karman himself, represents it as two infinite rows of vortices: one set of line vortices of strength Γ at $z = na$, and another set of strength $-\Gamma$ at $z = (n + \frac{1}{2})a + ib$, with $n = 0, \pm 1, \pm 2, \ldots$. Find the speed at which such a pattern of vortices moves, assuming that each line vortex moves at the local flow velocity due to everything other than itself.



☆ **Problem 2** A smoke ring in still air travels slowly in the direction perpendicular to the plane of the ring (see the figure). In such a ring the smoke particles rotate around the hollow toroidal axis of the doughnut (toroid) in the directions indicated with the arrows. What makes smoke rings travel through the air? Which way will the smoke ring in the diagram travel?

Two smoke rings can chase one another, the trailing ring accelerating and shrinking while the leading ring slows down and expands. The smaller ring catches up with the larger one and passes through. Then the roles are reversed and the process is repeated! A fascinating show, but how do we explain it?



✷ **Problem 3** An infinite cylinder of radius a rotates in a viscous, incompressible fluid with angular velocity Ω. Show that the vortex of a following form:

$$\mathbf{u} = \frac{\Omega a^2}{r}\mathbf{e}_\theta \qquad r \geq a \tag{1}$$

is in this case an exact solution of the Navier-Stokes equation satisfying the boundary conditions. Show that there is a nonzero torque exterted on such a cylinder by a fluid and find the value of this torque. Next, find a mistake in the following reasoning:
"Navier-Stokes eqution for viscous, incompressible fluid

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g} + \eta\nabla^2\mathbf{u}$$

can be transformed, using a formula

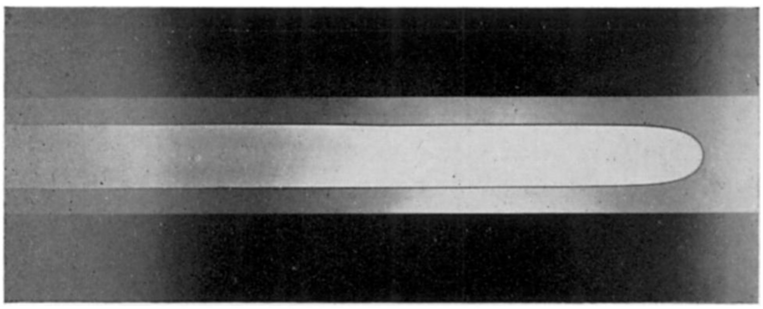
$$\nabla^2\mathbf{u} = \text{grad div}\mathbf{u} - \text{rot rot}\mathbf{u},$$

into the form

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g} - \eta \text{ rot rot}\mathbf{u}.$$

However the flow (1) is irrotational (check!), thus the viscous term in the Navier-Stokes equations vanishes; therefore the viscous forces are zero; and so the torque on the cylinder is zero."

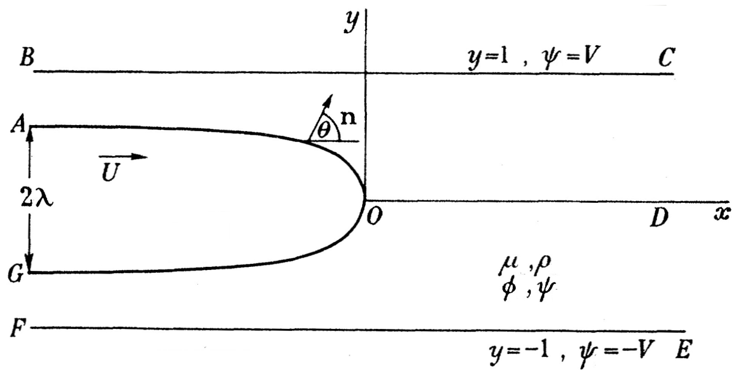
⁑**Problem 4** When a less viscous fluid is injected into a layer of a more viscous fluid confined in a narrow gap between two parallel plates, a viscous fingering instability occurs. Remarkably, in the long-time limit, the interface transforms itself into what is known as a Saffman-Taylor finger, which propagates in the direction of the flow without any further change in shape:



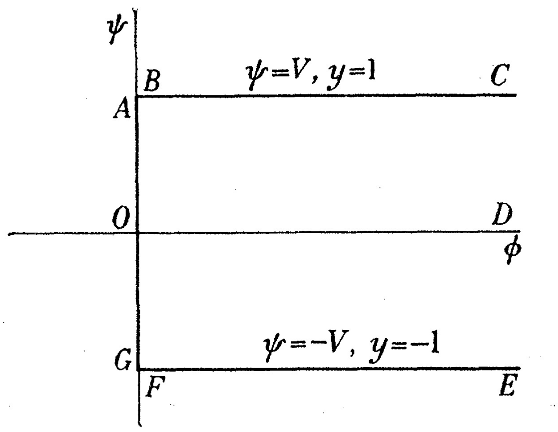
Treating the flow in the system as a two-dimensional, irrotational potential flow, it is shown that the shape of this finger is described by the following functional form:

$$x = \frac{W(1-\lambda)}{2\pi}\log\left(\frac{1+\cos(2\pi y/\lambda W)}{2}\right) \tag{2}$$

where W is the width of the cell, and λW is the width of the finger, as illustrated in the figure below.



Assume that the viscosity of the invading fluid (e.g., air) is significantly lower compared to the viscosity of the fluid in the cell, such that the velocity potential inside the finger can be considered constant. Assume that the constant velocity of the finger is equal to U, while the velocity of the receding fluid at $x \to \infty$ is equal to V, with $V = \lambda U$ (demonstrate this). Hint: You might find the so-called complex hodograph method useful. Instead of finding the velocity potential Φ and stream function Ψ as functions of $z = x + iy$, find z as a function of $w = \Phi + i\Psi$, treating Φ and Ψ as coordinates. In these coordinates, the geometry of the problem is, in fact, much simpler, as illustrated in the figure below (substantiate this)



Please bring the solution of not more than one problem to the first tutorial in the New Year

Merry Christmas and a Happy New Year! 🎄🎄🎄

Maciej Lisicki & Piotr Szymczak