Problem 4

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$$\bar{u} = \frac{Q}{2\pi r}\hat{e}_r \qquad \nabla\cdot\bar{u} = \frac{1}{r}\partial_r(r u_r) = 0$$

and $\nabla \times \bar{u} = 0$ except for $\underline{r = 0}$

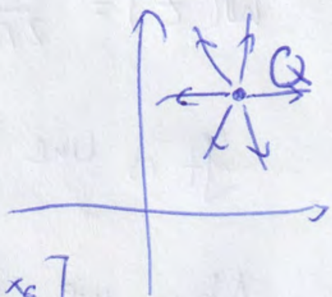
Lets find the potential:

$$\begin{cases} \partial_r \phi = u_r \\ \frac{1}{r}\partial_\theta \phi = u_\theta \end{cases} \Rightarrow \phi = \frac{Q}{2\pi}\ln r$$

$$\begin{cases} \frac{1}{r}\partial_\theta \psi = u_r \\ -\partial_r \psi = u_\theta \end{cases} \Rightarrow \psi = \frac{Q}{2\pi}\theta$$

$$\omega = \phi + i\psi = \frac{Q}{2\pi}\ln r e^{i\theta} = \frac{Q}{2\pi}\ln z$$

Lets generalize for: $\bar{u} = \frac{Q}{2\pi|\bar{r}-\bar{r}_0|^2}(\bar{r}-\bar{r}_0)$



~~$\Rightarrow \omega = \frac{Q}{2\pi}\ln(z - z_0)$~~

$$\bar{u} = \frac{Q}{2\pi|\bar{r}-\bar{r}_0|^2}\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} = \frac{Q}{2\pi}\frac{1}{(x-x_0)^2+(y-y_0)^2}\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

$$\begin{cases} \partial_x \phi = u_x \\ \partial_y \phi = u_y \end{cases} \Rightarrow \phi = \frac{Q}{2\pi}\ln\left[(x-x_0)^2+(y-y_0)^2\right]^{1/2}$$

$$\begin{cases} \partial_y \psi = u_x \\ -\partial_x \psi = u_y \end{cases} \Rightarrow \psi = \frac{Q}{2\pi}\operatorname{arctg}\left(\frac{y-y_0}{x-x_0}\right)$$

$$\omega = \phi + i\psi = \frac{Q}{2\pi}\left(\ln|z-z_0| + i\operatorname{arctg}\left(\frac{\operatorname{Im}(z-z_0)}{\operatorname{Re}(z-z_0)}\right)\right) = \frac{Q}{2\pi}\ln(z-z_0)$$

$$W(Z) = \omega(z) = \omega(f^{-1}(Z)) = \frac{Q}{2\pi}\ln\left(f^{-1}(Z) - f^{-1}(Z_0)\right)$$

Lets do conformal mapping :) $Z = f(z)$

We get ~~by inverse~~ from TLO: $f'(z_0) \neq 0 \Rightarrow \exists_{\Theta \ni z_0} : f^{-1}(Z)$ [illegible] is well defined $Z \in \Theta$

Then locally at this Θ:

$$\ln\left(f^{-1}(Z) - f^{-1}(Z_0)\right) \approx \ln\left((f^{-1})'(Z_0)(Z - Z_0)\right) = *$$

and $(f^{-1})' = \left[f' \circ f^{-1}\right]^{-1}$ ← TFU

Thus $* = \ln\left(f'(z_0)(Z - Z_0)\right) = \ln f'(z_0) + \ln(Z - Z_0)$

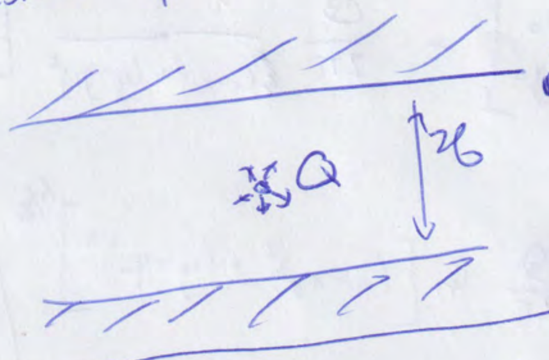
So everything:

constant we can ignore

$$W(Z) = \frac{Q}{2\pi}\left(\ln f'(z_0) + \ln(Z - Z_0)\right) \equiv \frac{Q}{2\pi}\ln(Z - Z_0)$$

It is line source of strength Q! (locally)

Now we consider flow:



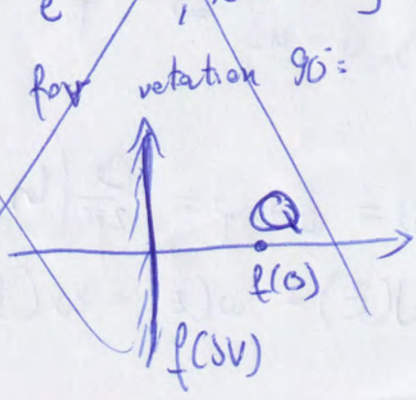
$w \neq$ [illegible]

2. With mapping $Z = e^{\alpha z}$ we get for boundary

$$\partial V = \{x \pm bi \,;\, x \in \mathbb{R}\} \xmapsto{f} \{e^{\alpha x} e^{\pm \alpha b i},\, x \in \mathbb{R}\} = f(\partial V)$$

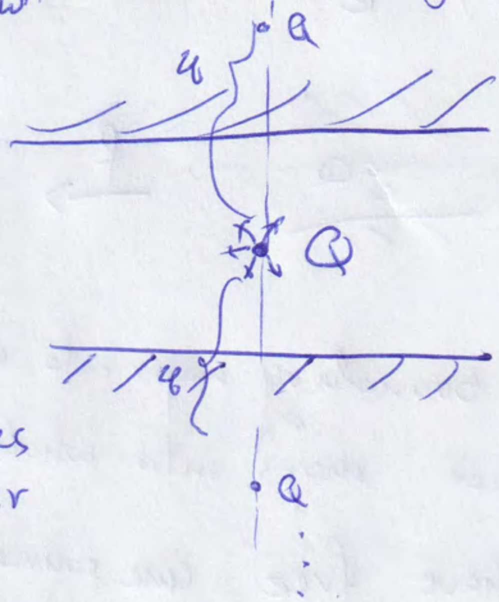
It is a straight line passing through (0,0) for rotation 90°:

$\alpha b = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{2b}$ we get



Problem 4 cd

a) lets solve the problem with method of images



To get appropriate boundary conditions we need to add infinite number of line sources 2b apart from each other

BC: $u_y|_{\partial V} = 0$

So the potential will be:

$$w(z) = \sum_{n=-\infty}^{\infty} \frac{Q}{2\pi} \ln(z - 2bni)$$ ← we nee to sum and renormalize it

~~$w(z) = \frac{Q}{2\pi} \sum_{n=-\infty}^{\infty} \ln\left(\frac{z}{2bni} - i\right) + \ln$~~

$$w(z) = \frac{Q}{2\pi}\left(\ln z + \sum_{n=1}^{\infty} \ln(z^2 + 4n^2b^2)\right) =$$

$$= \frac{Q}{2\pi}\left[\ln z + \sum_{n=1}^{\infty} \ln\left(\frac{z^2}{4n^2b^2} + 1\right) + 2\ln(2nb)\right] =$$

infinite constant, we drop as renormalisation

$$= \frac{Q}{2\pi} \ln\left[z \prod_{n=1}^{\infty}\left(\frac{z^2}{\pi^2 n^2}\left(\frac{\pi}{2b}\right)^2 + 1\right)\right] = \frac{Q}{2\pi} \ln\left[\frac{\pi z}{2b} \prod_{n=1}^{\infty}\left(\frac{\left(\frac{\pi z}{2b}\right)^2}{\pi^2 n^2} + 1\right)\right] =$$

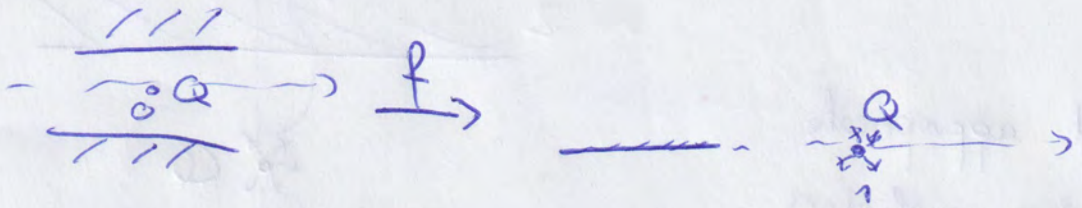
we add constant

infinite series representation of sh

$$= \frac{Q}{2\pi} \ln\left(\operatorname{sh}\left(\frac{\pi z}{2b}\right)\right)$$ ← WE GET OUR POTENTIAL

b) If we solve with conformal transformation:

$$f(z) = e^{\frac{\pi z}{b}}$$

$\Rightarrow$ 

So boundary maps into a line, and flow region maps into whole space, then we have free line source (we know it is line source from lemma we proved earlier).

So $W(f) = \frac{Q}{2\pi} \ln(f-1) \overset{!}{=} w(z)$

$$\Rightarrow w(z) = W(f(z)) = \frac{Q}{2\pi}\ln(f(z)-1) = \frac{Q}{2\pi}\ln\left(e^{\frac{\pi z}{b}}-1\right) =$$

$$= \frac{Q}{2\pi}\ln\left(\mathrm{sh}\left(\frac{\pi z}{2b}\right)\right) + \underbrace{\frac{Q}{2\pi}\frac{\pi z}{b}}_{\frac{Qz}{2b}}$$

$\frac{Qz}{2b}$ ← Strange addition of uniform flow to the right?

I can't find problem with this solution.
What am I missing?

Problem 4 odd

Yang Gandayh

Solution: We didn't account for boundary conditions properly.

At $\pm\infty$ flow is non zero in the initial problem

We can observe that $+\infty \xrightarrow{f} +\infty$

but $\lim_{x\to\infty} (x+iy) \xmapsto{f} 0$ So we need sink of strength $\frac{Q}{4\pi}$ at 0 after the transformation to ~~take~~ have proper boundary conditions!

Then

$$W(f) = \frac{Q}{2\pi}\ln(f-1) - \frac{Q}{4\pi}\ln f =$$

$$= \frac{Q}{2\pi}\ln\left(f^{1/2} - f^{-1/2}\right) \stackrel{!}{=} \omega(z)$$

$$\Rightarrow \omega(z) = W(f(z)) = \frac{Q}{2\pi}\ln\left(e^{\frac{\pi z}{2b}} - e^{-\frac{\pi z}{2b}}\right) =$$

$$= \frac{Q}{2\pi}\ln\left(\overset{\text{const term}}{2}\operatorname{sh}\left(\frac{\pi z}{2b}\right)\right) \equiv \frac{Q}{2\pi}\ln\left(\operatorname{sh}\left(\frac{\pi z}{2b}\right)\right)$$

Same solution as for method of images ♡

We solved correctly with both methods!