Hydrodynamics and Elasticity - Problem Sheet 8

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Problem 4

We will prove the following identity:

$$F_x - iF_y = \frac{i\rho}{2} \oint \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z$$

The complex potential w is defined as $w = \phi + i\psi$, and it's differential is $\mathrm{d}w = \mathrm{d}\phi + i\mathrm{d}\psi$. At the body boundary, the velocity must be tangent to the surface, this means that the contour C is a streamline of the flow. The stream function ψ is constant along streamlines, so on C we have $\mathrm{d}\psi = 0$. This allows us to write $\mathrm{d}w = \mathrm{d}\phi = \mathrm{d}\overline{w}$. Using this fact we can transform the contour integral:

$$\oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z = \oint_C \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\overline{z}}{\mathrm{d}\overline{z}}\mathrm{d}z = \oint_C \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\overline{w}}{\mathrm{d}z}\frac{\mathrm{d}\overline{z}}{\mathrm{d}\overline{z}}\mathrm{d}z = \oint_C \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\overline{w}}{\mathrm{d}\overline{z}}\mathrm{d}\overline{z} = \oint_C \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 \mathrm{d}\overline{z} = \oint_C v^2 \mathrm{d}\overline{z} = \oint_C v^2 \left(\mathrm{d}x - i\mathrm{d}y\right)$$

For steady, incompressible fluid we can use the Bernoulli's law:

$$\frac{p}{\rho} + \frac{v^2}{2} = c = \mathrm{const} \implies v^2 = c - \frac{2p}{\rho}$$

Integrating the constant c over a closed contour will give a zero contribution, we will only consider the term with p.

$$\frac{i\rho}{2} \oint \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z = -\frac{i\rho}{2} \cdot \frac{2}{\rho} \oint_C p \left(\mathrm{d}x - i\mathrm{d}y\right) = -\oint_C p \left(\mathrm{d}y + i\mathrm{d}x\right) = -\oint_C p\mathrm{d}y - i\oint_C p\mathrm{d}x = F_x - iF_y \qquad \square$$

Where we have used the fact that force in x direction is the integral of pressure over y "surface", $-\oint_C p\mathrm{d}y$, and y force is the same for x surface, $\oint_C p\mathrm{d}x$. The signs of the integrals is dependent on the contour orientation. Here we choose the standard anti-clockwise direction to get a minus sign in x and plus in y to obtain the form of the formula from the theorem.