Problem 7.2 Mateusz Polakowski

During tutorials we have looked at the problem of an expanding bubble in an incompressible fluid. This problem is very similar. Since the fluid fills all space, ~~then its density remains constant~~ and is incompressible $\Rightarrow \rho = \text{const}$.

$\Rightarrow \bar{\nabla} \cdot \bar{v} = 0$

We expect its movement to be radial $\Rightarrow$

$$4\pi \int_0^R \bar{\nabla} \cdot \bar{r}\, r^2 dr = 4\pi R^2 v \Rightarrow \bar{\nabla} \cdot \bar{v} = \frac{1}{r^2}\frac{d}{dr}(r^2 v) \overset{!}{=} 0 \Rightarrow$$

$$v = \frac{F(t)}{r^2}$$

From Euler equation we have

$$\rho\left(\frac{\dot{F}(t)}{r^2} + \frac{F(t)}{r^2}\partial_r\left(\frac{F(t)}{r^2}\right)\right) = -\bar{\nabla} p \Rightarrow \frac{\dot{F}(t)}{r^2} - 2\frac{F^2(t)}{r^5} = -\frac{1}{\rho}\partial_r p$$

Integrating from the boundary of the bubble $R(t)$ to $\infty$ we get:

$$p_\infty - p(R(t)) = \rho \int_{R(t)}^{\infty} \left(2\frac{F^2(t)}{r^5} - \frac{\dot{F}(t)}{r^2}\right) dr = \rho\left(\frac{\dot{F}(t)}{R(t)} - \frac{1}{2}\frac{F^2(t)}{R^4(t)}\right)$$

And since $v(R(t)) = \frac{F(t)}{R^2(t)} \Rightarrow \dot{R} = \frac{F(t)}{R^2(t)}$

Pressure in the bubble is zero $\Rightarrow p(R(t)) = 0$. With this in mind, we have the proper solution obtained during the tutorials:

$$\dot{R} = \sqrt{\frac{2p_\infty}{3\rho}\left(\frac{R_0^3}{R^3} - 1\right)} \Rightarrow dt = \sqrt{\frac{3\rho}{2p_\infty}} \frac{dR}{\sqrt{\left(\frac{R_0}{R}\right)^3 - 1}}$$

What is left, is to integrate this expression with proper limits:

$$\tau = \sqrt{\frac{3\rho}{2p_\infty}} \int_0^{R_0} \frac{dR}{\sqrt{\left(\frac{R_0}{R}\right)^3 - 1}} = \sqrt{\frac{3\rho}{2p_\infty}} \int_0^{R_0} \frac{R^{3/2} dR}{\sqrt{R_0^3 - R^3}} = \sqrt{\frac{3\rho}{2p_\infty}} \int_0^{R_0} \frac{R^{3/2} dR}{R_0^{3/2}\sqrt{1 - \left(\frac{R}{R_0}\right)^3}} =$$

$$= \left| \begin{array}{l} x = \frac{R}{R_0} \\ dR = R_0 dx \end{array} \right| = \sqrt{\frac{3\rho}{2p_\infty}} \int_0^1 \frac{R_0 x^{3/2} dx}{\sqrt{1 - x^3}} = \sqrt{\frac{3\rho}{2p_\infty}} R_0 \frac{\sqrt{\pi}\,\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}$$