Hydrodynamics and Elasticity 2023/2024

Sheet 6 - solution of problem 4

Problem 4 **Roof collapse problem**: A beam of length l is simply supported† and has initial small deflection $\eta_0 \sin \frac{\pi z}{l}$. The beam is subjected to a system of transverse forces distributed along the length of the beam and proportional to the value of deflection: $K = \alpha\eta$. (Such a force may be caused by rain water collecting on top of a deflected roof). Find the limiting value of the coefficient α at which a beam of a given length l remains in stable state.

† A beam is referred to as simply supported if both its ends are hinged, and one of the hinges can freely slide in the axial direction.

Solution The equation for the deflection of the beam is $\frac{d^4\eta}{dz^4} = \frac{K}{EI}$. The value of K is proportional to the total deflection — the sum of the initial and current one, i.e., $\frac{d^4\eta}{dz^4} = \frac{\alpha}{EI}\left(\eta + \eta_0 \sin \frac{\pi z}{l}\right)$ The solution of this nonhomogeneous equation is

$$\eta(z) = C_1 \sin kz + C_2 \cos kz + C_3 \sinh kz + C_4 \cosh kz + \frac{\alpha l^4}{\pi^4 EI - \alpha l^4}\eta_0 \sin \frac{\pi z}{l}$$

with $k^4 = \frac{\alpha}{EI}$. The boundary conditions $\eta(0) = \eta(l) = 0$, $\eta''(0) = \eta''(l) = 0$ determine the constants: $C_1 = C_2 = C_3 = C_4 = 0$. The solution has then the form

$$\eta(z) = \frac{\alpha l^4}{\pi^4 EI - \alpha l^4}\eta_0 \sin \frac{\pi z}{l}$$

The amplification coefficient of the amplitude grows with α. At $\alpha = \alpha_{cr} = \frac{\pi^4 EI}{l^4}$, the amplification coefficient becomes infinite and the roof loses stability.