Problem 1: Solution The function $y(x)$ determining the deflection satisfies the following differential equation

$$\frac{\partial^4 y}{\partial x^4} = \frac{1}{EI} qg,$$

which can be directly integrated to yield

$$y(x) = \frac{qg}{24EI} x^4 + ax^3 + bx^2 + cx + d.$$

The constants a, b, c, d are found from boundary conditions at both fixed ends

$$y(0) = 0, \qquad y'(0) = 0, \qquad y(l) = 0, \qquad y'(l) = 0.$$

Thus we find

$$y(x) = -\frac{qg}{24EI} x^2 (l - x)^2$$

The maximal deflection is in the middle $y_{max} = y(l/2)$ and has the value

$$y_{max} = \frac{qg}{384} \frac{l^4}{EI}.$$

The torque is found from the Euler-Bernoulli law

$$M = -EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2},$$

and takes the value of

$$M(0) = -\frac{1}{12} qgl^2,$$

The force is given by $F = \frac{\mathrm{d}M}{\mathrm{d}x}$, so at the ends we have

$$R = \frac{qgl}{2}.$$

Notw This problem can be solved also by starting from the Euler-Bernoulli law

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{M(x)}{EI}$$

where $M(x)$ is an external force moment acting on a cross-section of the rod at x. With the way the ends of the rod are fixed, there are reaction forces R and moments M_0, and by symmetry they must be of the same value at both ends. They contribute to the total force moment $M(x)$, just as the gravity of the rod does. The resulting total distribution of force moment has the form

$$M(x) = -\int_0^x qg\xi \mathrm{d}\xi + Rx + M_0 = -\frac{1}{2} qgx^2 + Rx + M_0.$$

From this we find the deflection of the beam as

$$y(x) = \frac{1}{EI} \left[-\frac{1}{24} qgx^4 + \frac{1}{6} Rx^3 + \frac{1}{2} M_0 x^2 + Cx + C_0 \right].$$

The constants R, M_0, C, C_0 are determined from the terminal boundary conditions:

$$y(0) = 0, \qquad y'(0) = 0, \qquad y(l) = 0, \qquad y'(l) = 0.$$

We thus find

$$y(x) = -\frac{qg}{24EI} x^2 (l - x)^2, \qquad y_{max} = \frac{qg}{384} \frac{l^4}{EI}$$

And the unknown force and moment are:

$$R = \frac{1}{2} qgl, \qquad M_0 = -\frac{1}{12} qgl^2.$$