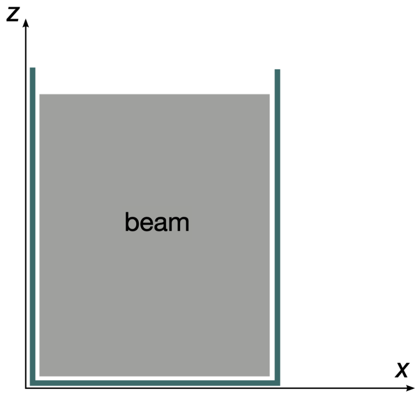
Hydrodynamics and Elasticity 2023/2024

Sheet 5 Problem 2 Solution

Problem 2 Consider an infinitely long (along the y axis) elastic beam with a rectangular cross-section of height h in a gravitational field $\boldsymbol{g} = (0, 0, -1)$. The beam is placed in a container, whose walls are perfectly rigid and perfectly slippery. The plane $z = 0$ is the bottom of the container. The walls allow the beam to move along their surfaces but do not allow motion in the direction perpendicular to their plane.

Find the stresses and deformation of the beam under its own weight. The density of the beam material is ρ_0. Sketch the shape of the deformed beam and the stress distribution. Is there a characteristic length scale associated with such a deformation?



By symmetry, we expext the deformation to be of the form

$$\boldsymbol{u} = u_z(z)\hat{\boldsymbol{e}}_z,$$

so that the only nonvanishing component of strain is $E_{zz} = \partial_z u_z$. From Hooke's law we thus find the stresses

$$T_{xx} = T_{yy} = \lambda E_{zz}, \quad T_{zz} = (\lambda + 2\mu)E_{zz}.$$

The Cauchy equilibrium equation is

$$\frac{\partial T_{zz}}{\partial z} = \rho_0 g.$$

Since the top surface of the beam is free, for $z = h$ we must have $T_{zz} = 0$. We readily obtain

$$T_{zz} = -\rho_0 g(h - z),$$

which means that the pressure $p_z = -T_{zz}$ is positive and increases with the depth $h - z$, just as in a containes of fluid. At each point, it balances the weight of the material above. This is expected, because there are no shear stresses which would distribute the vertical load. However, there is horizontal pressure $p_y = p_z\lambda/(\lambda + 2\mu)$, which is smaller than vertical pressure but still positive. The deformation

$$E_{zz} = -\frac{\rho_0 g}{\lambda + 2\mu}(h - z)$$

is negative, which corresponds to the material being compressed. The characteristic lenght scale is

$$D = \frac{\lambda + 2\mu}{\rho_0 g}.$$

Integrating the strain tensor with the condition $u_z = 0$ for $z = 0$ (rigid bottom), we get

$$u_z = -\frac{h^2 - (h - z)^2}{2D}$$

The deformation is thus negative, maximal on the top surface, and increases quadratically with height.

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