Proof: We have

$$W'(z) = u - iv = q\mathrm{e}^{-i\chi}$$

so $|\mathbf{u}| = q$. So, Bernoulli (no g, no t) says

$$p/\rho + \tfrac{1}{2}q^2 = p_0/\rho$$

where p_0 is a constant background pressure which exists in the absence of the flow. Hence

$$p = p_0 - \tfrac{1}{2}\rho q^2.$$

and the force on C is given by

$$\mathbf{F} = -\int_C p\hat{\mathbf{n}}\,\mathrm{d}s = \int_C \rho q^2 \hat{\mathbf{n}}\,\mathrm{d}s$$

since $\int_C p_0\hat{\mathbf{n}}\,\mathrm{d}s = \int_C \nabla p_0 dxdy = 0$ by the divergence theorem and the fact that p_0 is a constant.

By definition, the flow velocity is everywhere parallel to the boundary C and letting $\chi(s)$ denote the angle that C makes to the positive x-axis as a function of arclength gives

$$\frac{dx}{ds} = \cos\chi, \quad \frac{dy}{ds} = \sin\chi, \qquad \Rightarrow \qquad dz = dx + idy = \mathrm{e}^{i\chi}ds.$$

Also, by geometrical considerations $\hat{\mathbf{n}} = (y_s, -x_s) = (\sin\chi, -\cos\chi)$. So if we write $\mathbf{F} = (F_x, F_y)$ in terms of its components and define a **complex force** $F = F_x - iF_y$ then the force from above can be written out as

$$\begin{aligned} F = F_x - iF_y = -\int_C p(\sin\chi + i\cos\chi)\,\mathrm{d}s &= -i\int_C p\mathrm{e}^{-i\chi}\,\mathrm{d}s \\ &= i\tfrac{1}{2}\rho\int_C (q^2\mathrm{e}^{-2i\chi})\,\mathrm{e}^{i\chi}\,\mathrm{d}s \\ &= i\tfrac{1}{2}\rho\int_C (q\mathrm{e}^{-i\chi})^2\,\mathrm{d}z = i\tfrac{1}{2}\rho\int_S (W'(z))^2\,\mathrm{d}z \end{aligned}$$

6.8 Method of Images: Flows next to cylinders

Theorem: Suppose $f(z)$ is a complex potential in the absence of a cylinder with no singularities in $|z| < a$. Then

$$\boxed{W(z) = f(z) + \overline{f}(a^2/z)} \qquad (45)$$

is the complex potential representing a flow in the presence of a cylinder on $|z| = a$.

This is called the **Milne-Thompson circle theorem**.

Proof: On $|z| = a$, $z\overline{z} = a^2$ and so $a^2/z = \overline{z}$. Hence

$$\overline{f}(a^2/z) = \overline{f}(\overline{z}) = \overline{f(z)}$$

Thus, on $|z| = a$, $W(z) = f(z) + \overline{f(z)}$ and $\Im\{W\} = \psi = 0$. The streamline may be replaced by rigid boundary and no new singularities have emerged in $|z| > a$.

E.g. 6.5: Choose $f(z) = Uz$ (§6.2, uniform flow). Then (45) gives us

$$W(z) = Uz + U\frac{a^2}{z}$$

I.e. stream plus horizontal dipole of strength $\mu = -2\pi U a^2$.

Note: Exactly the flow found in §5.3.1 for flow past a cylinder.

Using Blasius, the complex force is

$$F_x - iF_y = \frac{1}{2}i\rho \int_C U^2 \left(1 - \frac{a^2}{z^2}\right) \, \mathrm{d}z = \frac{1}{2}i\rho U^2 \int_C \left(1 - 2\frac{a^2}{z^2} + \frac{a^4}{z^4}\right) dz = 0$$

since there are no simple poles inside C. We already had this result from §5.8 when U is constant.

6.8.1 Problem: Vortex outside a cylinder

Find the complex potential for a point vortex outside a cylinder and determine its motion.

In absence of cylinder, point vortex at z_0 is $f(z) = \frac{-i\Gamma}{2\pi}\log(z - z_0)$. With a cylinder, radius $a < |z_0|$ (45) gives

$$\begin{aligned} W(z) &= -\frac{i\Gamma}{2\pi}\log(z - z_0) + \frac{i\Gamma}{2\pi}\log\left(\frac{a^2}{z} - \overline{z_0}\right) \\ &= -\frac{i\Gamma}{2\pi}\left\{\log(z - z_0) - \log\left(\frac{1}{z}(-\overline{z_0})\left(z - \frac{a^2}{\overline{z_0}}\right)\right)\right\} \\ &= -\frac{i\Gamma}{2\pi}\left\{\log(z - z_0) + \log(z) - \log\left(z - \frac{a^2}{\overline{z_0}}\right) - \log(-\overline{z_0})\right\} \end{aligned}$$

The 2nd and 3rd terms are images at the origin and an inverse point to z_0 and the last term is a constant and can be ignored, because constants do not affect the flow velocities which are determined by derivatives.

(i) Motion of vortex

The velocity field at $z = z_0$ is due to the image vortices, or

$$u - iv = W'(z_0) - f'(z_0) = -\frac{i\Gamma}{2\pi}\left\{\frac{1}{z_0} - \frac{1}{z_0 - a^2/\overline{z_0}}\right\}$$

Better to work in polar coordinates, so let $z_0 = r_0(t)\mathrm{e}^{i\theta_0(t)}$ track the position of the vortex whence

$$\begin{aligned} q\mathrm{e}^{-i\chi} = -\frac{i\Gamma}{2\pi}\left\{\frac{1}{r_0\mathrm{e}^{i\theta_0}} - \frac{r_0\mathrm{e}^{-i\theta_0}}{r_0^2 - a^2}\right\} &= \frac{i\Gamma}{2\pi}\mathrm{e}^{-i\theta_0}\left(\frac{a^2}{r_0(r_0^2 - a^2)}\right) \\ &= \frac{\Gamma a^2}{2\pi r_0(r_0^2 - a^2)}\mathrm{e}^{-i(\theta_0 - \pi/2)}. \end{aligned}$$

Thus, the speed of the point vortex is $\Gamma a^2/2\pi r_0(r_0^2 - a^2)$ and its direction is at right angles to its position. Remembering the representation for velocity in polars:

$$\mathbf{u} = \dot{r_0}\hat{\mathbf{r}} + r_0\dot{\theta}_0\hat{\boldsymbol{\theta}} = -\frac{\Gamma a^2}{2\pi r_0(r_0^2 - a^2)}\hat{\boldsymbol{\theta}}$$

(Mech 1) means

$$\frac{dr_0}{dt} = 0, \qquad \frac{d\theta_0}{dt} = -\frac{\Gamma a^2}{2\pi r_0^2(r_0^2 - a^2)}$$

and the first equation integrates to $r_0(t) = r_0(0)$, a constant (initial radial distance to the vortex). The second integrates to

$$\theta_0(t) = \theta_0(0) - \frac{\Gamma a^2 t}{2\pi r_0^2(0)(r_0^2(0) - a^2)}.$$

Thus, the vortex moves at constant angular velocity in a circle around the cylinder.

(ii) Force on cylinder

From the Blasius formula and our definition of $W(z)$ we have

$$\begin{aligned}
F_x - iF_y &= \tfrac{1}{2}i\rho \int_C \left(-\frac{i\Gamma}{2\pi}\right)^2 \left(\frac{1}{z - z_0} + \frac{1}{z} - \frac{1}{z - a^2/\bar{z}_0}\right)^2 dz \\
&= -\frac{i\rho\Gamma^2}{8\pi^2}\int_{|z|=a} \left(\frac{1}{(z - z_0)^2} + \frac{1}{z^2} + \frac{1}{(z - a^2/\bar{z}_0)^2}\right. \\
&\qquad \left. + \frac{2}{z(z - z_0)} - \frac{2}{(z - z_0)(z - a^2/\bar{z}_0)} - \frac{2}{z(z - a^2/\bar{z}_0)}\right) dz
\end{aligned}$$

We can use Cauchy's Residue Theorem to evaluate the integral. The first 3 terms in the integral are poles of order 2 and don't contribute. Also, z_0 is outside $|z| = a$ but $a^2/\bar{z}_0$ is inside $|z| = a$ and only simple poles inside will count. So we get

$$F_x - iF_y = -\frac{i\rho\Gamma^2}{8\pi^2}(2\pi i)\left(\frac{2}{-z_0} - \frac{2}{(a^2/\bar{z}_0 - z_0)} - \frac{2}{a^2/\bar{z}_0} - \frac{2}{-a^2/\bar{z}_0}\right).$$

The last two terms cancel and the others combine as

$$F_x - iF_y = \frac{\rho\Gamma^2}{2\pi}\left(-\frac{1}{z_0} - \frac{\bar{z}_0}{a^2 - |z_0|^2}\right) = \frac{\rho\Gamma^2 a^2}{2\pi z_0(|z_0|^2 - a^2)}.$$

Writing $z_0 = r_0 e^{i\theta_0}$ shows that the force is of magnitude

$$\frac{\rho\Gamma^2 a^2}{2\pi r_0(r_0^2 - a^2)}$$

and is in the direction of θ_0.

E.g. if $z_0 = b > a$, a real number, then $F_y = 0$ and $F_x > 0$ and the cylinder feels a force towards the vortex.